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Schemes

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Introduction

This semester project is an introduction to algebraic geometry. It follows closely [2, Chapter II, 1 and 2] but with an emphasis on the categorical approach. All the proofs have been left out, but there are solutions to some exercises. The first part introduces all the categorical notions which will be needed to define a sheaf, such as functors, natural transformations and direct limits...

The second part follows [2, Chapter II, 1 Sheaves] and contains the solution of some of the exercises. It’s an introduction to sheaf theory with some examples. Sheaves are a way to collect local data all into one place, and then see how it behave when we focus on a single point. The notion of sheaf is necessary to define a scheme.

The third part introduce the notion of ringed spaces and schemes. It follows [2, Chapter II, 1 Sheaves] and also contains some solution to the exercises. Schemes were introduced by Alexander Grothendieck in order to broaden the notion of algebraic variety. A scheme can be seen as a topological space together with a sheaf of commutative rings.

1 A categorical introduction

Definition 1.1 (Category). A category $\mathcal{C}$ is a class of objects $\text{Ob}(\mathcal{C})$ together with a class of morphisms (or arrows) $\text{Mor}_\mathcal{C}(A, B)$ for any objects $A, B \in \text{Ob}(\mathcal{C})$ such that:

- there is a composition law:

$$\text{Mor}_\mathcal{C}(A, B) \times \text{Mor}_\mathcal{C}(B, C) \longrightarrow \text{Mor}_\mathcal{C}(A, C)$$

$$(f, g) \mapsto g \circ f;$$

- the composition law is associative;

- for every object $C \in \text{Ob}(\mathcal{C})$, there exists a morphism $1_C \in \text{Mor}_\mathcal{C}(C, C)$ called identity morphism, such that for every morphism $f \in \text{Mor}_\mathcal{C}(A, B)$,

$$1_B \circ f = f = f \circ 1_A.$$

Remark. We often write $A \in \mathcal{C}$ instead of $A \in \text{Ob}(\mathcal{C})$, and $f : A \to B$ instead of $f \in \text{Mor}_\mathcal{C}(A, B)$.

Example.

1. The class of sets together with the class of functions is a category denoted $\mathbb{Set}$.

2. The class of abelian groups together with the class of morphisms of groups is a category denoted $\mathbb{Ab}$.
3. The class of commutative rings together with the class of morphisms of ring is a category denoted $\mathbf{CRing}$.

4. The class of topological spaces together with the class of continuous maps is a category denoted $\mathbf{Top}$.

5. For a topological space $X$, we define a category $\mathbf{Top}(X)$, whose objects are the open subsets of $X$ and where the only morphisms are the inclusion maps. Thus $\text{Mor}(V, U) = \emptyset$ if $V \nsubseteq U$ and $\text{Mor}(V, U) = \{V \hookrightarrow U\}$ if $V \subseteq U$, where $U$ and $V$ are open subsets of $X$.

**Definition 1.2** (Dual category). Let $\mathcal{C}$ be a category. The dual category $\mathcal{C}^{\text{op}}$ is defined by:

- $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$;
- for each $A, B \in \text{Ob}(\mathcal{C}^{\text{op}})$, $\text{Mor}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Mor}_{\mathcal{C}}(B, A)$ and $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$.

**Definition 1.3** (Initial object). Let $\mathcal{C}$ be a category. An initial object of $\mathcal{C}$ is an object $I \in \mathcal{C}$ such that for every $X \in \mathcal{C}$, there exists a unique morphism from $I$ to $X$.

*Example.* In the category of commutative rings, $\mathbb{Z}$ is an initial object.

**Definition 1.4** (Final object). Let $\mathcal{C}$ be a category. A final object of $\mathcal{C}$ is an object $F \in \mathcal{C}$ such that for every $X \in \mathcal{C}$, there exists a unique morphism from $X$ to $F$.

*Example.* In the category of groups, $0$ is an initial and final object.

**Definition 1.5** (Isomorphism). Let $\mathcal{C}$ be a category. An isomorphism is a morphism $f : A \to B$ in $\mathcal{C}$ such that there exists a morphism $f^{-1} : B \to A$ in $\mathcal{C}$ with $f \circ f^{-1} = 1_A$ and $f^{-1} \circ f = 1_B$.

**Definition 1.6** (Zero morphism). Let $\mathcal{C}$ be a category. $\mathcal{C}$ has zero morphisms if for every $A, B \in \mathcal{C}$ there exists a morphism $0_{A, B} : A \to B$ of $\mathcal{C}$ such that for every morphism $f : R \to S$, $g : U \to V$ of $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow^{0_{R,U}} & & \downarrow^{0_{S,V}} \\
U & \xrightarrow{g} & V
\end{array}
$$

commutes.

*Example.* A category $\mathcal{C}$ with a final and initial object $I$ has zero morphisms. For $A, B \in \mathcal{C}$, $O_{A,B} : A \to B$ is the composition between $A \to I$ and $I \to B$, where $I$ is first seen as a final object and then as an initial object.
Definition 1.7 (Kernel). Let \( \mathcal{C} \) be a category with zero morphisms, and let \( f : X \to Y \) be a morphism of \( \mathcal{C} \). A kernel of \( f \) is a pair \((K, k)\) where \( K \in \mathcal{C} \) and \( k : K \to X \) is a morphism of \( \mathcal{C} \) such that:

- \( f \circ k \) is the zero morphism from \( K \) to \( Y \);
- for any morphism \( k' : K' \to X \) such that \( f \circ k' \) is the zero morphism, there exists a unique morphism \( u : K' \to K \) making the following diagram commute.

\[
\begin{array}{ccc}
K & \xrightarrow{k} & X \\
\downarrow{\exists u} & \searrow{0_{K,Y}} & \downarrow{f} \\
K' & \xleftarrow{k'} & Y \\
\end{array}
\]

Example. In the category \( \text{Ab} \), \( K \) is the usual kernel of morphism of groups, and \( k \) is the inclusion morphism.

Definition 1.8 (Product). Let \( \mathcal{C} \) be a category and let \( A, B \in \text{Ob}(\mathcal{C}) \). A product of \( A \) and \( B \) is an object of \( \mathcal{C} \), denoted \( A \times B \), with a pair \( p_A : A \times B \to A \) and \( p_B : A \times B \to B \) of morphisms, such that for any pair \( f : C \to A \) and \( g : C \to B \) there exists a unique morphism \( u : C \to A \times B \) making the diagram commute.

\[
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow{\exists u} & \downarrow{p_A} & \downarrow{p_B} \\
A \times B & \xleftarrow{g} & B \\
\end{array}
\]

Example. In the category of sets, \( \text{Set} \), the product is the cartesian product of two sets.

Definition 1.9 (Coproduct). Let \( \mathcal{C} \) be a category and let \( A, B \in \text{Ob}(\mathcal{C}) \). A coproduct of \( A \) and \( B \) is an object of \( \mathcal{C} \), denoted \( A \oplus B \), with a pair \( i_A : A \to A \oplus B \) and \( i_B : B \to A \oplus B \) of morphisms, such that for any pair \( f : A \to C \) and \( g : B \to C \) there

\[
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow{\exists u} & \downarrow{i_A} & \downarrow{i_B} \\
A \oplus B & \xleftarrow{g} & B \\
\end{array}
\]
exists a unique morphism \( u : A \oplus B \to C \) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & A \oplus B & \xleftarrow{i_B} & B \\
\downarrow f & & \swarrow \exists! u & & \searrow g \\
C
\end{array}
\]

commute.

Remark. The coproduct is sometimes called the **direct sum**. The coproduct of \( A \) and \( B \) is sometimes denoted \( A \sqcup B \).

Example. In the category of sets, the coproduct is the disjoint union of two sets.

**Definition 1.10** (Functor). Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. A **covariant functor** \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \) is a mapping that:

- associates to each object \( C \in \mathcal{C} \) an object \( F(C) \in \mathcal{D} \);
- associates to each morphism \( f : A \to B \) of \( \mathcal{C} \) a morphism \( F(f) : F(A) \to F(B) \) of \( \mathcal{D} \);

satisfying the following two conditions:

- \( F(1_C) = 1_{F(C)} \) for all \( C \in \mathcal{C} \);
- \( F(f \circ g) = F(f) \circ F(g) \) for all morphisms \( g : A \to B \) and \( f : B \to C \) of \( \mathcal{C} \).

Remark. A **contravariant functor** \( F : \mathcal{C} \to \mathcal{D} \), is a functor \( F : \mathcal{C}^{\text{op}} \to \mathcal{D} \).

Example. The identity functor \( 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \) is defined by:

- \( 1_{\mathcal{C}}(A) = A \) for all \( A \in \mathcal{C} \);
- \( 1_{\mathcal{C}}(f) = f \) for all morphisms \( f : A \to B \) of \( \mathcal{C} \).

**Definition 1.11** (Natural transformation). Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, and let \( F \) and \( G \) be functors from \( \mathcal{C} \) to \( \mathcal{D} \). A **natural transformation** \( \eta \) from \( F \) to \( G \) is a mapping that associates to every object \( A \in \mathcal{C} \) a morphism \( \eta_A : F(A) \to G(A) \) of \( \mathcal{D} \) called the **component** of \( \eta \) at \( A \), such that for every morphism \( f : A \to B \) of \( \mathcal{C} \), the following diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow \eta_A & & \downarrow \eta_B \\
G(A) & \xleftarrow{G(f)} & G(B)
\end{array}
\]

commutes.
Remark.

- If both $F$ and $G$ are contravariant functors, the horizontal arrows in this diagram are reversed.
- If, for every object $A$ in $\mathcal{C}$, the morphism $\eta_A$ is an isomorphism in $\mathcal{D}$, then $\eta$ is said to be a natural isomorphism.
- If $\eta : F \to G$ and $\varepsilon : G \to H$ are natural transformations between functors $F, G, H : \mathcal{C} \to \mathcal{D}$, then we can compose them to get a natural transformation $\varepsilon \eta : F \to H$. This is done "componentwise": $(\varepsilon \eta)_X = \varepsilon_X \circ \eta_X$. This "vertical composition" of natural transformation is associative and has an identity, and allows one to consider the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ itself as a category.

Example. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The identity natural transformation from $F$ to $F$, denoted $1_F$, is the natural transformation such that for every $X$ in $\mathcal{C}$, the component of $1_F$ at $X$ is $1_F(X) : F(X) \to F(X)$.

Definition 1.12 (Counit-unit adjunction). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant counit-unit adjunction between $\mathcal{C}$ and $\mathcal{D}$ consists of two functors

$$F : \mathcal{D} \to \mathcal{C},$$
$$G : \mathcal{C} \to \mathcal{D},$$

and two natural transformations

$$\varepsilon : FG \to 1_\mathcal{C},$$
$$\eta : 1_\mathcal{D} \to GF,$$

respectively called the counit and the unit of the adjunction, such that the compositions

$$F \xrightarrow{F \eta} FGF \xrightarrow{\varepsilon F} F,$$
$$G \xrightarrow{\eta G} GFG \xrightarrow{G \varepsilon} G,$$

are the identity transformations $1_F$ and $1_G$ on $F$ and $G$ respectively.

Remark.

1. The equations

$$\varepsilon_{F(Y)} \circ F(\eta_Y) = 1_{F(Y)} \quad \forall Y \in \mathcal{D}$$
$$G(\varepsilon_X) \circ \eta_{G(X)} = 1_{G(X)} \quad \forall X \in \mathcal{C}$$

are called the first and second counit-unit equations.
2. If $F$ and $G$ are both contravariant functors, the first natural transformation, $\varepsilon$ is defined from $1_C$ to $FG$, and the adjunction is called a contravariant counit-unit adjunction.

**Proposition 1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be two adjoint functors and let $\varepsilon : FG \to 1_C$ and $\eta : 1_D \to GF$ be a counit-unit adjunction. Then for all $X \in \mathcal{C}$ and for all $Y \in \mathcal{D}$,

$$\text{Mor}_\mathcal{C}(F(Y), X) \cong \text{Mor}_\mathcal{D}(Y, G(X)).$$

**Proof.** [1, Chapter 3, theorem 3.1.5] For each $f : F(Y) \to X$ and for each $g : Y \to G(X)$, define

$$\Phi(f) = G(f) \circ \eta_Y$$
$$\Psi(g) = \varepsilon_X \circ F(g).$$

Since $\varepsilon$ and $\eta$ are natural transformations so are $\Phi$ and $\Psi$.

We compute the first composition:

$$\Psi(\Phi(f)) = \varepsilon_X \circ FG(f) \circ F(\eta_Y)$$
$$= f \circ \varepsilon_{F(Y)} \circ F(\eta_Y)$$
$$= f \circ 1_{F(Y)} = f \quad (\varepsilon_{F(Y)} \circ F(\eta_Y) = 1_{F(Y)}).$$

Hence $\Psi \Phi$ is the identity transformation.

Dually, we compute the second composition:

$$\Phi(\Psi(g)) = G(\varepsilon_X) \circ GF(g) \circ \eta_Y$$
$$= G(\varepsilon_X) \circ \eta_{G(X)} \circ g$$
$$= 1_{G(X)} \circ g = g \quad (G(\varepsilon_X) \circ \eta_{G(X)} = 1_{G(X)}).$$

Hence $\Phi \Psi$ is the identity transformation, so $\Phi$ is a natural isomorphism with inverse $\Phi^{-1} = \Psi$. \qed

**Definition 1.13** (Directed set). A directed set is a pair $(I, \leq)$ such that:

- $I \neq \emptyset$;
- $\leq$ is a reflexive and transitive binary relation i.e., $\leq$ is a preorder;
- for each $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$ (i.e., $k$ is an upper bound of $\{i, j\}$).

**Remark.** A directed set is a category whose objects are the elements of $I$ and for every $i, j \in I$, $\text{Mor}(i, j) = \{(i, j)\}$ if $i \leq j$, otherwise $\text{Mor}(i, j) = \emptyset$, with the following composition: $(i, j) \circ (j, k) = (i, k)$ for all $i \leq j \leq k$ in $I$. 7
Definition 1.14 (Direct system). Let $(I; \leq)$ be a directed set, and let $\mathcal{C}$ be a category. A direct system is a functor from $I$ to $\mathcal{C}$.

Remark. A direct system is denoted $\{X_i, f_{i,j}\}_{i,j \in I}$, where $\{X_i\}_{i \in I}$ is a family of objects and $\{f_{i,j}\}_{i,j \in I}$ is a family of morphism such that $f_{i,j} : X_i \to X_j$, for all $i \leq j$ in $I$.

Definition 1.15 (Direct limit). Let $\mathcal{C}$ be a category. Let $\{X_i, f_{i,j}\}_{i,j \in I}$ be a direct system of objects and morphisms of $\mathcal{C}$. The direct limit of this system is an object $X \in \mathcal{C}$ together with morphisms $\{\phi_i\}_{i \in I}$, $\phi_i : X_i \to X$, such that $\phi_i = \phi_j \circ f_{i,j}$ for all $i \leq j$ in $I$. Moreover $\{X, \phi_i\}_{i \in I}$ is universal in the sense that for any other object $Y \in \mathcal{C}$ together with morphisms $\{\psi_i\}_{i \in I}$, $\psi_i : X_i \to Y$ such that $\psi_i = \psi_j \circ f_{i,j}$ for all $i \leq j$ in $I$, there exists a unique morphism $u : X \to Y$ making the following diagram commute.

Remark. The direct limit may not exist in an arbitrary category.

2 Sheaves

2.1 Theory

Definition 2.1 (Presheaf). Let $X$ be a topological space. A presheaf $\mathcal{F}$ of abelian groups on $X$ is a contravariant functor from $\mathsf{Top}(X)$ to $\mathsf{Ab}$, such that $\mathcal{F}(\emptyset) = 0$.

Remarks.

(a) Thus to prove that $\mathcal{F}$ is a preheaf, one should verify the following properties:

0. $\mathcal{F}(\emptyset) = 0$.

1. For each open set $U$ of $X$, $\mathcal{F}(U)$ is an abelian group.

2. For each open sets $V \subseteq U$, there is a morphism of groups res$_{V,U}$ from $\mathcal{F}(U)$ to $\mathcal{F}(V)$. 


Moreover, the morphisms satisfy the following properties:

- for each open set $U$ of $X$, $\text{res}_{U,U} = 1_{\mathcal{F}(U)}$;
- for all open sets $W \subseteq V \subseteq U$, $\text{res}_{W,U} = \text{res}_{W,V} \circ \text{res}_{V,U}$.

(b) Let $V \subseteq U$ be two open sets of $X$. The image by $\mathcal{F}$ of the morphism $V \hookrightarrow U$ of $\text{Top}(X)$, denoted $\text{res}_{V,U}$, is called restriction morphism. We often write $s|_V$ instead of $\text{res}_{V,U}(s)$, where $s \in \mathcal{F}(U)$.

(c) The elements of $\mathcal{F}(U)$ are called the sections of the presheaf $\mathcal{F}$ over the open set $U$. The elements of $\mathcal{F}(X)$ are called the global sections.

(d) We can define sheaves of sets, or sheaves of commutative rings, by changing the target of the functor to an appropriate category.

\textbf{Definition 2.2} (Sheaf). Let $X$ be a topological space. A sheaf of abelian groups $\mathcal{F}$ on $X$ is a presheaf of groups on $X$ which satisfies the following conditions.

3. (Uniqueness) If $U$ is an open subset of $X$, if $\{V_i\}_{i \in I}$ an open covering of $U$, and if $s \in \mathcal{F}(U)$ is a section such that $s|_{V_i} = 0$ for all $i \in I$, then $s = 0$.

4. (Glueing local sections) If $U$ is an open subset of $X$, if $\{V_i\}_{i \in I}$ an open covering of $U$, $\{s_i\}_{i \in I}$ are such that $s_i \in \mathcal{F}(V_i)$ and $\text{res}_{V_i \cap V_j, U}(s_i) = \text{res}_{V_i \cap V_j, U}(s_j)$, then there exists $s \in \mathcal{F}(U)$ such that $\text{res}_{V_i, U}(s) = s_i$ (s is unique by condition 3).

\textbf{Example.}

1. For any open subset $U \subseteq \mathbb{R}$, let $\mathcal{F}(U)$ be the set of bounded functions from $U$ to $\mathbb{R}$. The restriction maps are the usual restrictions of functions. Then $\mathcal{F}$ is a presheaf, but $\mathcal{F}$ is not a sheaf. Consider the following open covering of $\mathbb{R}$, $V_i = \{x \in \mathbb{R} : |x| < i\}$ for all $i \in \mathbb{N}$, the identity function $1_{V_i} : x \mapsto x$ is bounded for every $i \in \mathbb{N}$ but the identity is not bounded on $\mathbb{R}$, and thus the glueing property is not satisfied.

2. Let $X$ be a topological space. For any open subset $U$ of $X$, let $\mathcal{F}(U) = C^0(U, \mathbb{R})$ the set of continuous map from $U$ to $\mathbb{R}$. The restrictions map are the usual restriction of functions. Then $\mathcal{F}$ is a sheaf.

It is clear that $\mathcal{F}$ is presheaf. Let’s prove that $\mathcal{F}$ is sheaf. Let $U$ be an open subset of $X$ and let $\{V_i\}_{i \in I}$ be an open covering of $U$.

(a) Let $f$ be an element of $C^0(U, \mathbb{R})$ such that $f|_{V_i} = 0$ for all $i \in I$. Let $s \in U$, in particular there exists $j \in I$ such that $s \in V_j$. As $f|_{V_j} = 0$, we have $f(s) = f_{V_j}(s) = 0$, hence $f = 0$. 


(b) Let \( \{f_i\}_{i \in I} \) be a family of maps such that \( f_i \in C^0(V_i, \mathbb{R}) \) for all \( i \in I \) and \( f_{ij}|_{V_i \cap V_j} = f_{ij}|_{V_i \cap V_j} \) for all \( i, j \in I \). We define \( f \) by \( f(x) = f_i(x) \) if \( x \in V_i \). As \( f_{ij}|_{V_i \cap V_j} = f_{ij}|_{V_i \cap V_j} \) for all \( i, j \in I \), \( f \) is well-defined. Moreover as \( f_i|_{V_i} = f_i \) is continuous for all \( i \in I \), and as \( \{V_i\}_{i \in I} \) is an open covering of \( U \), clearly \( f \in C^0(U, \mathbb{R}) \).

**Definition 2.3** (Stalk). Let \( \mathcal{F} \) be a presheaf on a topological space \( X \) and let \( x \) be a point in \( X \). The **stalk** \( \mathcal{F}_x \) of \( \mathcal{F} \) at \( x \) is the direct limit of the groups \( \mathcal{F}(U) \) for all the open neighbourhoods \( U \) of \( x \). Thus \( \mathcal{F}_x = \lim_{x \in U} \mathcal{F}(U) \).

**Remark.**

1. The direct system is given by open sets of \( X \) containing \( x \), together with the restriction morphisms, using the following preorder: \( U \leq V \) if \( V \subseteq U \).

2. The elements of the stalk are called **germs**.

3. It’s not hard to see that an element of \( \mathcal{F}_x \) is represented by a pair \( < U, s > \), where \( U \) is an open neighbourhood of \( x \), and \( s \in \mathcal{F}(U) \). Moreover, two such pairs \( < U, s > \) and \( < V, t > \) are equal in \( \mathcal{F}_x \) if and only if there exists an open neighbourhood \( W \) of \( x \), such that \( W \subseteq U \cap V \) and \( s|_W = t|_W \).

**Example.** Let \( \mathcal{F} \) be the sheaf of continuous maps from \([0, 1]\) (with the induce topology) to \( \mathbb{R} \). Let \( x \in [0, 1] \). The stalk \( \mathcal{F}_x \) is \( C([0, 1], \mathbb{R})/\sim \), where \( \sim \) is the following equivalence relation:

\[
f \sim g \iff \exists U \subseteq [0, 1], x \in U \text{ open}, f|_U = g|_U.
\]

**Definition 2.4** (Morphism of presheaves). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two presheaves on a topological space \( X \). A **morphism of presheaves** \( \phi : \mathcal{F} \to \mathcal{G} \), is a natural transformation from \( \mathcal{F} \) to \( \mathcal{G} \).

**Remark.**

1. Thus a morphism of presheaves consists in, for every open subset \( U \subseteq X \), a morphism of groups \( \phi(U) : \mathcal{F}(U) \to \mathcal{G}(U) \), such that for every open subset \( V \subseteq U \) and for every \( s \in \mathcal{F}(U) \), \( \phi(U)(s)|_V = \phi(V)(s|_V) \).

2. The class of presheaves of abelian groups on a topological space \( X \) together with the class of morphisms of presheaves is a category, denoted \( \text{Prsh}(X) \).

3. If \( \mathcal{F} \) and \( \mathcal{G} \) are sheaves, a morphism of presheaves from \( \mathcal{F} \) to \( \mathcal{G} \) is called a **morphism of sheaves**.

4. The class of sheaves of abelian groups on a topological space \( X \) together with the class of morphisms of sheaves is a category, denoted \( \mathbf{Ab}(X) \).
Proposition 2.1. Let $\mathcal{F}, \mathcal{G}$ be sheaves, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then for all $x \in X$, the following morphism

$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$

$< U, s > \mapsto < U, \varphi(U)(s)>,$

where $U$ is an open subset of $X$ and $s \in \mathcal{F}(U)$, is well-defined.

Proof. Let $U$ and $V$ be two open subsets of $X$. Let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ such that $< U, s > = < V, t >$. Thus there exists $W \subseteq U \cap V$ such that $s|_W = t|_W$. Let’s show that $\varphi_x(< U, s >) = \varphi_x(< V, t >)$:

$$
(\varphi(U)(s))|_W = \varphi(W)(s|_W) \quad (\varphi \text{ is a morphism of presheaves})
$$

$$
= \varphi(W)(t|_W) \quad (s|_W = t|_W)
$$

$$
= (\varphi(V)(t))|_W. \quad (\varphi \text{ is a morphism of presheaves})
$$

Thus $\varphi_x(< U, s >) = \varphi_x(< V, t >)$, and $\varphi_x$ is well-defined. $\square$

Proposition 2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space $X$, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then $\varphi$ is an isomorphism of sheaves if and only if the induced map on the stalks is an isomorphism of groups for all $x \in X$.

Proof. See [2, Chapter II, Proposition 1.1]. $\square$

Definition 2.5 (Sheaf associated to a presheaf). Let $\mathcal{F}$ be a presheaf on a topological space $X$. The sheaf associated to $\mathcal{F}$ is a sheaf $\mathcal{F}^+$ together with a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$, such that for every sheaf $\mathcal{G}$, and for every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism of sheaves $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\
\downarrow{\varphi} & & \downarrow{\psi} \\
\mathcal{G} & & \exists! \psi
\end{array}
$$

commutes.

Proposition 2.3. Let $\mathcal{F}$ be a presheaf on $X$. Then the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ exists and is unique up to isomorphism. Moreover, $\theta_x : \mathcal{F}_x \to \mathcal{F}_x^+$ is an isomorphism for every $x \in X$.

Proof. See [3, Chapter 2, Proposition 2.15] $\square$
Definition 2.6 (Presheaf kernel). Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space $X$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. The \textit{presheaf kernel} of $\varphi$ is the presheaf $U \mapsto \ker(\varphi(U))$.

Proposition 2.4. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Then $\ker(\varphi) : U \mapsto \ker(\varphi(U))$ is a sheaf.

Proof. First we show that $\ker(\varphi)$ is a presheaf. As $\varphi(U)$ is a morphism of groups, it is clear that $\ker(\varphi(U))$ is an abelian group for all open subsets $U$ of $X$. Let $V \subseteq U$ be an open subset. We define $\rho_{V,U} : \ker(\varphi(U)) \to \ker(\varphi(V))$ by $\rho_{V,U}(s) = s|_{V}$ for all $s \in \ker(\varphi(U))$. Using the fact that $\varphi$ is a morphism of presheaves, we see that $\rho_{V,U}$ is well-defined, i.e., $\text{im}(\rho_{V,U}) \subseteq \ker(\varphi(V))$. We now show that $\ker(\varphi)$ is a presheaf:

0. $\ker(\varphi)(\emptyset) = \ker(\varphi(\emptyset)) = \ker(0) = \{0\}$
1. Let $s \in \ker(\varphi(U))$, $\rho_{U,U}(s) = s|_{U} = s = \text{id}_{\ker(\varphi(U))}(s)$.
2. Let $W \subseteq V \subseteq U$ be open subsets of $X$ and $s \in \ker(\varphi(U))$. Then

$$\rho_{W,V} \circ \rho_{V,U}(s) = \rho_{W,V}(s|_{V}) = (s|_{V})|_{W} = s|_{W} \quad \text{(because $\mathcal{F}$ is presheaf)}$$

$= \rho_{W,U}(s) \quad \text{(because $s \in \ker(\varphi(U))$)}$

We now show that $\ker(\varphi)$ is a sheaf. Let $U$ be an open subset of $X$ and $\{V_{i}\}_{i \in I}$ an open covering of $U$.

3. Let $s \in \ker(\varphi(U))$ such that $\rho_{V_{i,U}}(s) = 0$ for all $i \in I$. As $\ker(\varphi(U)) \subseteq \ker(U)$, $s \in \mathcal{F}(U)$ and as $\rho_{V_{i,U}}(s) = \text{res}_{V_{i,U}}(s) = 0$ for all $i \in I$, and since $\mathcal{F}$ is a sheaf, using the local identity condition, we have $s = 0$.

4. Let $s \in \ker(\varphi(U))$ and $\{s_{i}\}_{i \in I}$ be a family of sections such that $s_{i} \in \ker(\varphi(V_{i}))$ for all $i \in I$ and $\rho_{V_{i} \cap V_{j},U}(s_{i}) = \rho_{V_{i} \cap V_{j},U}(s_{j})$ for all $i, j \in I$. Since $\mathcal{F}$ is a sheaf, by the glueing condition there exists $s \in \mathcal{F}(U)$ such that $s|_{V_{i}} = s_{i}$ for all $i \in I$. We now show that $s$ is in $\ker(\varphi(U))$,

$$\varphi(V_{i})(s|_{V_{i}}) = \varphi(V_{i})(s_{i}) = 0 \quad \forall i \in I$$

$$= (\varphi(U)(s))|_{V_{i}} = 0 \quad \forall i \in I.$$ 

And thus as $\varphi(U)(s) \in \mathcal{G}(U)$ and as $\mathcal{G}$ is a sheaf, by the local identity condition $\varphi(U)(s) = 0$, i.e., $s \in \ker(\varphi(U))$. \qed

Remark. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Then the presheaf kernel of $\varphi$ is called \textit{kernel} and denoted $\ker \varphi$.  

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Definition 2.7 (Presheaf image). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two presheaves on a topological space \( X \). Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a morphism of presheaves. The presheaf image of \( \varphi \) is the presheaf defined by \( U \mapsto \text{im}(\varphi(U)) \).

Definition 2.8 (Image). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two sheaves on a topological space \( X \). Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a morphism of sheaves. The image of \( \varphi \), denoted \( \text{im}(\varphi) \), is the sheaf associated to the presheaf image of \( \varphi \).

Definition 2.9 (Injective morphism of sheaves). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two sheaves on a topological space \( X \). A morphism of sheaves \( \varphi : \mathcal{F} \to \mathcal{G} \) is injective if \( \ker(\varphi)(U) = 0 \) for all open subset \( U \subseteq X \).

Definition 2.10 (Surjective morphism of sheaves). Let \( \mathcal{F} \) and \( \mathcal{G} \) be two sheaves on a topological space \( X \). A morphism of sheaves \( \varphi : \mathcal{F} \to \mathcal{G} \) is surjective if \( \text{im}(\varphi) = \mathcal{G} \).

Remark. If \( \varphi : \mathcal{F} \to \mathcal{G} \) is surjective, the morphisms \( \varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U) \) need not to be surjective for all open subset \( U \subseteq X \).

Definition 2.11 (Exact sequence of sheaves). A sequence of sheaves and morphisms

\[
\cdots \to \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \to \cdots
\]

is exact if \( \ker \varphi_i = \text{im} \varphi_{i-1} \) for all \( i \).

Definition 2.12 (Direct image sheaf). Let \( X \) and \( Y \) be two topological spaces, let \( \mathcal{F} \) be a sheaf on \( Y \), and let \( f : X \to Y \) be a continuous map. The direct image sheaf, denoted \( f_* \mathcal{F} \), is the sheaf on \( Y \) defined by \( (f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V)) \) for all open subsets \( V \subseteq Y \).

Definition 2.13 (Inverse image sheaf). Let \( X \) and \( Y \) be two topological spaces, let \( \mathcal{G} \) be a sheaf on \( Y \), and let \( f : X \to Y \) be a continuous map. The inverse image sheaf, denoted \( f^{-1} \mathcal{G} \), is the sheaf associated to the presheaf on \( X \), defined by, \( U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V) \) for all open subsets \( U \subseteq X \).

Definition 2.14 (Restriction of sheaves). Let \( \mathcal{F} \) be a sheaf on a topological space \( X \). Let \( Z \subseteq X \) be an open set with the induced topology. If \( i : Z \to X \) is the inclusion map, the restriction of \( \mathcal{F} \) to \( Z \), denoted \( \mathcal{F}|_Z \), is the sheaf \( i^{-1} \mathcal{F} \). Thus \( \mathcal{F}|_Z \) is the sheaf \( U \mapsto \mathcal{F}(U) \) where \( U \) is an open subset of the topological space \( Z \).

2.2 Exercises

Exercise. [2, Chapter II, Exercise 1.2]

(a) Let \( \varphi : \mathcal{F} \to \mathcal{G} \) be a morphism of presheaves and \( x \in X \), then

\[
\ker \varphi_x = (\ker \varphi)_x.
\]

and

\[
\text{im} \varphi_x = (\text{im} \varphi)_x.
\]
(b) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves and $x \in X$. $\varphi$ is injective (respectively surjective) if and only if the induced map on the stalk $\varphi_x$ is injective (respectively surjective) for all $x \in X$.

(c) Show that a sequence $\ldots \to \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $x \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

**Solution.** (a) Let $< U, s > \in \mathcal{F}_x$ be an element of $\ker \varphi_x$. Then

$$\varphi_x(< U, s >) = < U, \varphi(U)(s) > = < X, 0 >,$$

so there exists an open set $W \subseteq U$ such that $(\varphi(U)(s))|_W = 0 = \varphi(W)(s|_W)$. Thus $s|_W \in \ker \varphi(W) = (\ker \varphi)(W)$ and then $< U, s > = < W, s|_W > \in (\ker \varphi)_x$.

Let $< U, s > \in \mathcal{F}_x$ be an element of $(\ker \varphi)_x$, i.e. $U$ is an open neighbourhood of $x$ and $s \in (\ker \varphi)(U)$. Thus $\varphi(U)(s) = 0$ and so,

$$\varphi_x(< U, s >) = < U, \varphi(U)(s) > = < U, 0 >.$$

Hence $< U, s > \in \ker \varphi_x$.

Let $< V, t > \in \text{im} \varphi_x$, i.e., $V$ is an open neighbourhood of $x$, $t \in \mathcal{G}(V)$ and there exists $< U, s > \in \mathcal{F}_x$ such that $\varphi_x(< U, s >) = < V, t >$. Thus there exists an open set $W \subseteq U \cap V$ such that $x \in W$ and $(\varphi(U)(s))|_W = t|_W$. Moreover as $\varphi$ is a morphism of sheaves, $(\varphi(U)(s)|_W = \varphi(W)(s|_W)$, and thus

$$< V, t > = < W, \varphi(W)(s|_W) >,$$

clearly $\varphi(W)(s|_W) \in \text{im} \varphi(W)$, thus $< V, t > \in (\text{im} \varphi)_x$ because the stalks of the presheaf image are in bijection with the stalks of the image.

Let $< V, t > \in (\text{im} \varphi)_x$, i.e. $V$ is an open neighbourhood of $x$ and $t \in \text{im}(\varphi(V))$. Thus there exists $s \in \mathcal{F}(V)$ such that $\varphi(V)(s) = t$, thus

$$< V, t > = < V, \varphi(V)(s) > = \varphi_x(< V, s >).$$

Thus $< V, t > \in \text{im} \varphi_x$.

(b) The result follows from (a) and the definition of an injective (respectively, surjective) morphism of sheaves.

(c) A sequence is exact if $\ker(\varphi^{i+1}) = \text{im}(\varphi^{i})$, hence the result follows from (a).
Exercise. [2, Chapter II, Exercise 1.5] Show that a morphism of sheaves is an isomor-
phism if and only if it is both injective and surjective.

Solution. It’s a consequence of the previous exercise part (b), and of the Proposition
2.2. □

Exercise. [2, Chapter II, Exercise 1.8] For any open subset $U \subseteq X$, show that the functor
$\Gamma$ from sheaves on $X$ to abelian groups (defined by $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$) is a left exact
functor, i.e., if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is an exact sequences of sheaves, then $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is an exact sequence of groups.

Solution. Let $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H}$ be an exact sequence of sheaves. We first
need to show that $\mathcal{G}(U)$ is injective for all open subsets $U \subseteq X$. But by definition of
an injective morphism of sheaves, $\ker(\mathcal{G}(U)) = \ker(\mathcal{G}(U)) = \{0\}$. Moreover, as the
sequence is exact, $\mathcal{G}(U) \circ \mathcal{G}(U) = 0$, thus $\mathcal{G}(U) \circ \mathcal{G}(U) = 0$, and thus $\im(\mathcal{G}(U)) \subseteq \ker(\mathcal{G}(U))$
for all open subsets $U \subseteq X$. As $\mathcal{G}$ is injective, for all open subsets $U \subseteq X$, $\mathcal{F}(U)$ and
$\im(\mathcal{G}(U))$ are isomorphic, thus there is a bijection between $\mathcal{F}$ and the presheaf image
of $\mathcal{G}$, $U \mapsto \im(\mathcal{G}(U))$, thus as $\mathcal{F}$ is a sheaf, the presheaf image of $\mathcal{G}$ is a sheaf too and
so $\im(\mathcal{G}(U)) = \im(\mathcal{G}(U))$ for all open subsets $U \subseteq X$. □

Exercise (Direct sum). [2, Chapter II, Exercise 1.9] Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a
topological space $X$. Show that the presheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ is a sheaf. It is called
the direct sum of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$. Show that it plays the role of
direct sum and of direct product in the category of sheaves of abelian groups on $X$.

Solution. We first show that $\mathcal{F} \times \mathcal{G} : U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$ is a sheaf and then that it
plays the role of direct product and of direct sum in the category of sheaves. Let $U$ be
an open subset of $X$. We define the restrictions maps as follow:

$$\rho_{V,U} : \mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{F}(V) \times \mathcal{G}(V) \quad (s,t) \mapsto (s_{|V}, t_{|V}).$$

Let $(s,t) \in \mathcal{F}(U) \times \mathcal{G}(U)$. We now verify the sheaf definition:

0. $\emptyset \mapsto \mathcal{F}(\emptyset) \times \mathcal{G}(\emptyset) = 0 \times 0 = 0$.

1. $\rho_{U,U}(s,t) = (s_{|U}, t_{|U}) = (s,t) = 1_{U \times U}(s,t)$.

2. Let $W \subseteq V \subseteq U$ be open subsets of $X$,

$$\rho_{W,V} \circ \rho_{V,U}(s,t) = ((s_{|V})_{|W}, (t_{|V})_{|W}) = (s_{|W}, t_{|W}) = \rho_{W,U}(s,t).$$

3. Let $\{V_i\}_{i \in I}$ be an open covering of $U$, and let $(s,t)$ be an element of $\mathcal{F}(U) \times \mathcal{G}(U)$
such that $\rho_{V_i,U}(s,t) = 0$ for all $i \in I$. Thus $\rho_{V_i,U}(s,t) = (s_{|V_i}, t_{|V_i}) = (0,0)$ for all
$i \in I$, and as $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $s = t = 0$.
4. Let \((s_i, t_i) \in \mathcal{F}(V_i) \times \mathcal{G}(V_i)\) for all \(i \in I\) such that \(\rho_{V_i \cap V_j, U}(s_i, t_i) = \rho_{V_i \cap V_j, U}(s_j, t_j)\) for all \(i, j \in I\). Thus, as

\[(s_i)_{V_i \cap V_j} = (s_j)_{V_i \cap V_j}\]

and as \(\mathcal{F}\) is a sheaf, there exists \(s \in \mathcal{F}(U)\) such that \(s|_{V_i} = s_i\) for all \(i \in I\). Using the same argument, there exists \(t \in \mathcal{G}(U)\) such that \(t|_{V_i} = t_i\) for all \(i \in I\). And thus \(\rho_{V_i, U}(s, t) = (s_i, t_i)\) for all \(i \in I\).

We now show that this sheaf plays the role of direct product. Let \(U \subseteq X\) be an open set. We define the following morphisms of sheaves

\[p_{\mathcal{F}} : \mathcal{F} \times \mathcal{G} \longrightarrow \mathcal{F}\]
\[p_{\mathcal{F}}(U) : (s, t) \longmapsto s,\]

and

\[p_{\mathcal{G}} : \mathcal{F} \times \mathcal{G} \longrightarrow \mathcal{G}\]
\[p_{\mathcal{G}}(U) : (s, t) \longmapsto t,\]

for every \((s, t) \in F(U) \times G(U)\). Let \(C\) be a sheaf, and let \(f : C \rightarrow \mathcal{F}\) and \(g : C \rightarrow \mathcal{G}\) be two morphisms of sheaves. We define \(u : C \rightarrow \mathcal{F} \times \mathcal{G}\) by

\[u(U)(c) = (f(U)(c), g(U)(c))\]

for all \(c \in C(U)\). Clearly \(p_{\mathcal{F}} \circ u = f\) and \(p_{\mathcal{G}} \circ u = g\). Let \(u' : C \rightarrow \mathcal{F} \times \mathcal{G}\) be a morphism of sheaves such that \(p_{\mathcal{F}} \circ u' = f\) and \(p_{\mathcal{G}} \circ u' = g\). As \(u'\) is determined by its components at \(\mathcal{F}\) and at \(\mathcal{G}\), clearly \(u' = u\) and thus by Definition 1.8, \(\mathcal{F} \times \mathcal{G}\) plays the role of direct product.

We now show that the direct product of two sheaves plays also the role of direct sum. Let \(U \subseteq X\) be an open set. We define the following morphisms of sheaves

\[i_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \times \mathcal{G}\]
\[i_{\mathcal{F}}(U) : s \longmapsto (s, 0),\]

and

\[i_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{F} \times \mathcal{G}\]
\[i_{\mathcal{G}}(U) : t \longmapsto (0, t),\]

for every \(s \in F(U)\) and every \(t \in G(U)\). Let \(C\) be a sheaf, and let \(f : \mathcal{F} \rightarrow C\) and \(g : \mathcal{G} \rightarrow C\) be two morphisms of sheaves. We define \(u : \mathcal{F} \times \mathcal{G} \rightarrow C\) by

\[u(U)(s, t) = f(U)(s) + g(U)(t)\]

for all \((s, t) \in \mathcal{F}(U) \times \mathcal{G}(U)\), where "+" is the operation law in the abelian group \(C(U)\). Clearly \(u \circ i_{\mathcal{F}} = f\), and \(u \circ i_{\mathcal{G}} = g\). Let \(u' : \mathcal{F} \times \mathcal{G} \rightarrow C\) be a morphism of
sheaves such that \( u' \circ i_x = f \) and \( u' \circ i_y = g \). Let \((s, t) \in F(U) \times G(U)\), as \(u'(U)\) is a morphism of groups:

\[
\begin{align*}
u'(U)(s, t) &= u'(U)((s, 0) + (0, t)) \\
&= u'(U)(s, 0) + u'(U)(0, t) \\
&= (u' \circ i_x)(U)(s, t) + u' \circ i_y(U)(s, t) \\
&= f(U)(s) + g(U)(t) \\
&= u(U)(s, t).
\end{align*}
\]

Hence \( u' = u \) and thus by Definition 1.9 \( \mathcal{F} \times \mathcal{G} \) plays also the role of direct sum.

**Exercise (Adjoint property).** [2, Chapter II, Exercise 1.18] Let \( X \) and \( Y \) be two topological spaces, and let \( f : X \to Y \) be a continuous map. Show that for any sheaves \( \mathcal{F} \) on \( X \) there is a natural map \( f^{-1} \mathcal{F} \to \mathcal{F} \), and for any sheaf \( \mathcal{G} \) on \( Y \) there is a natural map \( \mathcal{G} \to f_* f^{-1} \mathcal{G} \). Use these maps to show that there is a natural bijection of sets, for any sheaf \( \mathcal{F} \) on \( X \) and \( \mathcal{G} \) on \( Y \),

\[
\text{Mor}_{\mathsf{Ab}}(X) \to \text{Mor}_{\mathsf{Ab}}(Y)
\]

Hence we say that \( f^{-1} \) is a left adjoint of \( f_* \), and that \( f_* \) is a right adjoint of \( f^{-1} \).

**Solution.** In the following diagram,

\[
\begin{array}{ccc}
\text{Mor}_{\mathsf{Prsh}(X)}(f^{-1} \mathcal{G}, \mathcal{F}) & \longrightarrow & \text{Mor}_{\mathsf{Prsh}(Y)}(\mathcal{G}, f_* \mathcal{F}) \\
\downarrow & & \downarrow \\
\text{Mor}_{\mathsf{Ab}(X)}(f^{-1} \mathcal{G}, \mathcal{F}) & \longrightarrow & \text{Mor}_{\mathsf{Ab}(Y)}(\mathcal{G}, f_* \mathcal{F})
\end{array}
\]

the left vertical arrow is an isomorphism (by definition of the associated sheaf), thus if we prove that the top horizontal arrow is an isomorphism then the bottom one will be an isomorphism too. Thus we will only prove the adjunction in the case of presheaves, and so we will only consider morphisms of presheaves.

We define the following natural transformation,

\[
\varepsilon : \mathcal{F}_{\mathsf{Prsh}}(X) \to \mathcal{F}_{\mathsf{Prsh}}(X)
\]

\[
\varepsilon_\mathcal{F} : f^{-1} f_* \mathcal{F} \to \mathcal{F},
\]

such that for all open subsets \( U \subseteq X \),

\[
\varepsilon_\mathcal{F}(U) : (f^{-1} f_* \mathcal{F})(U) \to \mathcal{F}(U)
\]

\[
< f^{-1}(V), s > \longmapsto s_{|U},
\]

where \( V \) is an open subset of \( Y \) such that \( f(U) \subseteq V \), and \( s \in \mathcal{F}(f^{-1}(V)) \). This is well-defined as \( f(U) \subseteq V \) implies \( U \subseteq f^{-1}(V) \), and thus \( s_{|U} \) exists. Let \( < f^{-1}(V), s > \)
be in $f^{-1}f_*\mathcal{F}$, where $f(U) \subseteq V$ and $s \in \mathcal{F}(f^{-1}(V))$. Let $W \subseteq U$ be two open subsets of $X$. We now show that $\varepsilon_{\mathcal{F}}$ is compatible with the restriction morphisms:

$$(\varepsilon_{\mathcal{F}}(U)(< f^{-1}(V), s >))|_W = (s|_U)|_W$$

(by definition of $\varepsilon$)

$= s|_W$ (because $\mathcal{F}$ is a sheaf)

$= \varepsilon_{\mathcal{F}}(W)(< f^{-1}(V), s >|_W)$

$= \varepsilon_{\mathcal{F}}(W)(< f^{-1}(V), s >).$

Thus $\varepsilon_{\mathcal{F}}$ is a morphism of presheaves. We now show that for every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{F}'$, the following diagram

$$
\begin{array}{ccc}
\varepsilon_{\mathcal{F}} & \xrightarrow{f^{-1}f_*(\varphi)} & \varepsilon_{\mathcal{F}'} \\
| & | & | \\
\mathcal{F} & \xrightarrow{\varphi} & \mathcal{F}'
\end{array}
$$

commutes. Let $\varphi : \mathcal{F} \to \mathcal{F}'$ be a morphism of presheaves, let $V$ be an open subset of $Y$ such that $f(U) \subseteq V$ and let $s \in \mathcal{F}(f^{-1}(V))$, using, in order, the definition of $\varepsilon$, that $\varphi$ is a morphism of presheaves, and again the definition of $\varepsilon$, we obtain:

$$(\varepsilon_{\mathcal{F}'} \circ (f^{-1}f_*\varphi))(U)(s) = \varepsilon_{\mathcal{F}'}(U)((f^{-1}f_*\varphi)(U)(< f^{-1}(V), s >))$$

$$= \varepsilon_{\mathcal{F}'}(U)(< f^{-1}(V), (f_*\varphi)(V)(s) >)$$

$$= \varepsilon_{\mathcal{F}'}(U)(< f^{-1}(V), \varphi(f^{-1}(V))(s) >)$$

$$= \varphi(f^{-1}(V))(s)|_U$$

$$= \varphi(U)(\varepsilon_{\mathcal{F}}(U)(< f^{-1}(V), s >))$$

$$= (\varphi \circ \varepsilon_{\mathcal{F}})(U)(< f^{-1}(V), s >).$$

Thus the diagram is commutative.

We now define the following natural transformation,

$$\eta : \mathcal{P}(Y) \to \mathcal{P}(Y)$$

$$\eta_{\mathcal{G}} : \mathcal{G} \to f_*f^{-1}\mathcal{G},$$

such that for all open subsets $U \subseteq Y$,

$$\eta_{\mathcal{G}}(U) : \mathcal{G}(U) \to (f_*f^{-1}\mathcal{G})(U)$$

$$s \mapsto < U, s >.$$
This is well-defined since \( f(f^{-1}(U)) \subseteq U \). Let \( W \subseteq U \) be two open subsets of \( X \), and let \( s \in \mathcal{G}(U) \). We now show that \( \eta_\mathcal{G} \) is compatible with the restriction morphisms:

\[
\eta_\mathcal{G}(W)(s|_W) = < W, s|_W > = ( < U, s > )|_W.
\]

Thus \( \eta_\mathcal{G} \) is a morphism of presheaves. Using the same arguments as before, we show that for each morphism of presheaves \( \psi : \mathcal{G} \rightarrow \mathcal{G}' \), the following diagram

\[
\begin{array}{ccc}
\mathcal{G} & \rightarrow & \mathcal{G}' \\
\eta_\mathcal{G} & \downarrow & \eta_\mathcal{G}' \\
f_* f^{-1} \mathcal{G} & \xrightarrow{f_* f^{-1}(\psi)} & f_* f^{-1} \mathcal{G}'
\end{array}
\]

commutes.

We compute the first counit-unit equation, \( f_* \varepsilon \circ \eta_{f_*} \). The morphism of presheaves \( \eta_{f_*} \) is defined by:

\[
\eta_{f_*}(U) : (f_* \mathcal{F})(V) \rightarrow (f_* f^{-1} f_* \mathcal{F})(V)
\]

\[
s \mapsto < V, s >.
\]

Notice that \( (f_* f^{-1} f_* \mathcal{F})(V) = \lim_{f(f^{-1}(V)) \subseteq U} \mathcal{F}(f^{-1}(U)) \) and \( f(f^{-1}(V)) \subseteq V \). The morphism of presheaves \( f_* \varepsilon \mathcal{F} \) is defined by:

\[
f_* \varepsilon \mathcal{F}(V) : (f_* f^{-1} f_* \mathcal{F})(V) \rightarrow (f_* \mathcal{F})(V)
\]

\[
< f^{-1}(U), s > \mapsto s|_V.
\]

Notice that \( s|_V = s|_{f^{-1}(V)} \) where \( s \) is first seen as an element of \( f_* \mathcal{F}(V) \) (sheaf over \( Y \)) and then as an element of \( \mathcal{F}(f^{-1}(V)) \) (sheaf over \( X \)). Thus it is clear that \( f_* \varepsilon \circ \eta_{f_*} = 1_{f_*} \).

We now compute the second counit-unit equation, \( \varepsilon f^{-1} \circ f^{-1} \eta \). The morphism of presheaves \( f^{-1} \eta_\mathcal{G} \) is defined by:

\[
f^{-1} \eta_\mathcal{G}(U) : (f^{-1} \mathcal{G})(U) \rightarrow (f^{-1} f_* f^{-1} \mathcal{G})(U)
\]

\[
< V, s > \mapsto < V, s >.
\]

The first \( < V, s > \) is seen as an element of

\[
(f^{-1} \mathcal{G})(U) = \lim_{f(U) \subseteq V} \mathcal{G}(V)
\]
where \( f(U) \subseteq V \) and \( s \in \mathcal{G}(V) \). The second \( < V, s > \) is seen as an element of \( (f^{-1}f_*f^{-1}\mathcal{G})(U) = \lim_{f(U) \subseteq V} (f_*f^{-1}\mathcal{G})(V) \) as \( f(U) \subseteq V \) and \( s \in (f_*f^{-1}\mathcal{G})(V) = \lim_{f(f^{-1}(V)) \subseteq W} \mathcal{G}(W) \), because \( f(f^{-1}(V)) \subseteq V \). The morphism of presheaves \( \varepsilon_{f^{-1}\mathcal{G}} \) is defined by:

\[
\varepsilon_{f^{-1}\mathcal{G}}(U) : (f^{-1}f_*f^{-1}\mathcal{G})(U) \longrightarrow (f^{-1}\mathcal{G})(U) \quad < V, s > \longmapsto s|_U.
\]

Notice that \( f(U) \subseteq V \) implies \( s|_U \) exists. Thus it is clear that \( \varepsilon f^{-1} \circ f^{-1} \eta = 1_{f^{-1}} \).

Hence we have a counit-unit adjunction between \( \text{Prsh}(X) \) and \( \text{Prsh}(Y) \), thus by Proposition 1.1,

\[
\text{Mor}_{\text{Prsh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \text{Mor}_{\text{Prsh}(Y)}(\mathcal{G}, f_*\mathcal{F})
\]

for all \( \mathcal{F} \in \text{Prsh}(X) \) and for all \( \mathcal{G} \in \text{Prsh}(Y) \).

\[ \square \]

3 Schemes

In this section, a ring will stand for a commutative ring with unit, \( 1 \neq 0 \).

3.1 Theory

We first define the algebraic notions needed to study schemes.

**Definition 3.1** (Local ring). A commutative ring \( A \), is a local ring if \( A \) has exactly one maximal ideal, denoted \( m_A \).

**Definition 3.2** (Local morphism of local rings). Let \( A \) and \( B \) be local rings with maximal ideal \( m_A \) and \( m_B \) respectively. A morphism of rings \( g : A \rightarrow B \) is a local morphism if \( g^{-1}(m_B) = m_A \).

**Definition 3.3** (Spectrum of a ring). Let \( A \) be a commutative ring. The spectrum of \( A \), denoted \( \text{Spec} A \), is the set of all proper prime ideals of \( A \).

**Definition 3.4.** Let \( A \) be a commutative ring and let \( a \) be an ideal of \( A \). We define the subset \( V(a) \) \( \subseteq \text{Spec} A \) to be the set of all prime ideals which contain \( a \). For an element \( f \in A \) we define \( D(f) = \text{Spec} A - V((f)) \).

**Proposition 3.1.** Let \( A \) be a commutative ring.

1. If \( a \) and \( b \) are ideals of \( A \), then \( V(ab) = V(a) \cup V(b) \).
2. If \( \{a_i\}_{i \in I} \) is a family of ideals of \( A \), then \( V(\sum_{i \in I} a_i) = \bigcap_{i \in I} V(a_i) \).
3. $V(A) = \emptyset$ and $V((0)) = \text{Spec } A$.

Proof. See [2, Chapter II, Lemma 2.1].

Definition 3.5 (Zariski topology). Let $A$ be a commutative ring. We define the Zariski topology on $\text{Spec } A$ by taking the subsets of the form $V(a)$ to be the closed subsets, where $a$ is an ideal of $A$.

Remark.

1. Proposition 3.1 ensures that the Zarisky topology is well-defined.

2. The sets of the form $D(f)$, $f \in A$, are a base of open subsets of $\text{Spec } A$.

Definition 3.6 (Multiplicatively closed set). Let $A$ be a commutative ring. A multiplicatively closed set of $A$, is a subset $S \subseteq A$ verifying the following conditions:

- $1 \in S$ and $0 \not\in S$;
- $S$ is closed under multiplication, i.e., for all $x, y \in S$, $xy \in S$.

Example. Let $f \in A$, $f$ not nilpotent, then $\{1, f, f^2, \ldots \} = \{f^i\}_{i \in \mathbb{N}}$ is a multiplicatively closed set.

Definition 3.7 (Localization of a ring). Let $A$ be a commutative ring and let $S$ be a multiplicatively closed set of $A$. The localization of $A$ at $S$, denoted $S^{-1}A$, is the ring defined as:

$$S \times A / \sim,$$

where $\sim$ is the following equivalence relation:

$$(s_1, a_1) \sim (s_2, a_2) \iff \exists t \in S, \ t(s_2a_1 - s_1a_2) = 0.$$

Remark. An element $(s, a) \in S^{-1}A$ is often denoted $\frac{a}{s}$.

Example.

1. Let $p$ be a prime ideal of $A$, then $A - p$ is a multiplicatively closed set of $A$ and we denote $A_p = (A - p)^{-1}A$.

2. Let $f \in A$, $f$ not nilpotent, and let $S$ be the multiplicatively closed set $\{f^i\}_{i \in \mathbb{N}}$.

The localization of $A$ at $S$ is denoted $A_f$, and $A_f = S^{-1}A$.

Definition 3.8 (Sheaf of rings on $\text{Spec } A$). Let $A$ be a commutative ring. We define a sheaf of rings, denoted $\mathcal{O}_{\text{Spec } A}$, over $\text{Spec } A$. For an open set $U \subseteq \text{Spec } A$ we define $\mathcal{O}_{\text{Spec } A}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} A_p$ such that $s(p) \in A_p$ for each $p$, and such that $s$ is "locally the quotient of elements of $A$". To be precise, we require that for each $p \in U$, there is a neighbourhood $V$ of $p$, contained in $U$, and elements $a, f \in A$, such that for each $q \in V$, $f \not\in q$, and $s(q) = \frac{a}{f}$ in $A_q$.

Now it is clear that sums and products of such functions are again such functions, and that the element 1 which gives 1 in each $A_p$ is an identity. Thus $\mathcal{O}_{\text{Spec } A}(U)$ is a commutative ring. If $V \subseteq U$ are two open sets of $\text{Spec } A$, the natural restriction map of functions is a morphism of rings. It is then clear that $\mathcal{O}_{\text{Spec } A}$ is a presheaf. Finally, it is clear from the local nature of the definition that $\mathcal{O}_{\text{Spec } A}$ is a sheaf.
Proposition 3.2. Let $A$ be a commutative ring.

1. For any $p \in \text{Spec } A$, the stalk $(\mathcal{O}_{\text{Spec } A})_p$ is isomorphic to the local ring $A_p$.
2. For any element $f \in A$, the ring $\mathcal{O}_{\text{Spec } A}(D(f))$ is isomorphic to the localised ring $A_f$.
3. In particular $\mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong A$.

Proof. See [2, Chapter II, Proposition 2.2].

Definition 3.9 (Ringed space). A ringed space is a pair $(X, \mathcal{O}_X)$, where $X$ is a topological space, and $\mathcal{O}_X$ is a sheaf of commutative rings on $X$.

Definition 3.10 (Morphism of ringed spaces). Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves (of commutative rings on $Y$).

Definition 3.11 (Locally ringed space). A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that for all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Definition 3.12 (Morphism of locally ringed spaces). A morphism of locally ringed spaces is a morphism $(f, f^\#)$ of ringed spaces, such that for all $x \in X$, the induced map of local rings $f_{x,x}^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local morphism of local rings.

Definition 3.13 (Isomorphism of locally ringed spaces). An isomorphism of locally ringed spaces is a morphism of locally ringed spaces with a two-sided inverse. Thus a morphism of locally ringed spaces $(f, f^\#)$ is an isomorphism if and only if $f$ is a homeomorphism, and $f^\#$ is an isomorphism of sheaves.

Proposition 3.3. Let $A$ be a commutative ring. Then $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space.

Proof. Clear from Proposition 3.2.

Definition 3.14 (Affine scheme). An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to some $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ where $A$ is a commutative ring.

Definition 3.15 (Scheme). A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood $U$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Remark. The class of schemes together with the class of morphisms of schemes is a category denoted $\mathcal{S}ch$.

Example. Let $k$ be a field, then $\text{Spec } k$ is an affine scheme whose topological space consists of one point $(0)$ and whose sheaf consists only of the field $k$.  

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3.2 Exercises

Exercise. [2, Chapter II, Exercise 2.1] Let $A$ be a ring, let $X = \text{Spec} A$, let $f \in A$ (not nilpotent), and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\text{Spec}(A_f)$.

Solution. Let $f \in A$ be a non nilpotent element. First notice that $D(f)$ is the set of all prime ideals $p$ of $A$ such that $f \not\in p$. We define the following morphisms of rings:

\[
i : A \longrightarrow A_f
\]

\[
a \longmapsto \frac{a}{1}
\]

and

\[
\alpha : \text{Spec} A_f \longrightarrow \text{Spec} A
\]

\[
q \longmapsto i^{-1}(q).
\]

As $i^{-1}(q)$ is prime, $\alpha$ is well-defined. Moreover $\text{im} \alpha \subseteq D(f)$. Suppose (ad absurdum) that there exists $q \in \text{Spec} A_f$ such that $i^{-1}(q) \not\subseteq D(f)$, i.e. $f \in i^{-1}(q)$, thus \(\frac{a}{1} \in q\), hence,

\[
\frac{1}{1} = \frac{f}{f} = \frac{1}{1} f_n
\]

(as $f$ is not nilpotent and as $q$ is an ideal of $\text{Spec} A_f$)

is an element of $q$, and thus $q = A_f \not\in \text{Spec} A_f$.

We now show that the following map is a homeomorphism:

\[
\beta : D(f) \longrightarrow \text{Spec} A_f
\]

\[
p \longmapsto i(p)A_f,
\]

where $i(p)A_f$ is the ideal of $A_f$ generated by $i(p)$. We can verify that

\[
i(p)A_f = \left\{ \frac{a}{f^n} : n \in \mathbb{N}, a \in p \right\}
\]

is prime in $A_f$, so $\beta$ is well-defined. Let $p \in D(f)$, we compute $\beta \circ \alpha$:

\[
i^{-1}(i(p)A_f) = i^{-1}\left\{ \frac{a}{f^n} : n \in \mathbb{N}, a \in p \right\} \supseteq p.
\]

To show the other inclusion, let $b \in i^{-1}\left\{ \frac{a}{f^n} : n \in \mathbb{N}, a \in p \right\}$, thus \(\frac{b}{1} = \frac{n}{f^n}\), where $a \in p$ and $n \in \mathbb{N}$ thus $f^{n+m}b = f^m a$ where $m \in \mathbb{N}$ and as $f \not\in p$, $b \in p$. Thus $i^{-1}(i(p)A_f) = p$.

Let $q \in \text{Spec} A_f$. The other composition, $a \circ \beta$:

\[
i(i^{-1}(q))A_f = i\left( \left\{ a : \frac{a}{1} \in q \right\} \right) A_f
\]

\[
= \left\{ \frac{a}{1} : \frac{a}{1} \in q \right\} A_f \subseteq q
\]

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To show the other inclusion, let \( b = \frac{c}{m} \in q \). As \( q \) is an ideal of \( A_f \), \( f^m b = \frac{c}{m} \in q \). Thus \( i(i^{-1}(q)) A_f = q \). Moreover as \( \beta(V(a)) = V(i(a) A_f) \), where \( a \) is an ideal of \( A \) and as \( \beta^{-1}(V(b)) = V(i^{-1}(b)) \), where \( b \) is an ideal of \( A_f \), \( \beta \) is a homeomorphism.

We now show that \( \beta^\ast : \mathcal{O}_{\text{Spec} A_f} \to \beta_* \mathcal{O}_X|_{D(f)} \) is an isomorphism of sheaves. Using the stalks, we need to show that

\[
(\mathcal{O}_{\text{Spec} A_f})_q \cong (\beta_* \mathcal{O}_X|_{D(f)})_q, \quad \forall q \in \text{Spec} A_f.
\]

Let \( q \in \text{Spec} A_f \) and let \( p = \beta^{-1}(q) \). Using Proposition 3.2 (2.) and that \( \beta \) is a bijection, we have

\[
(\mathcal{O}_{\text{Spec} A_f})_q \cong (A_f)_q \cong (A_f)_{\beta(p)} \cong (A_f)_{\beta^{-1}(q)}.
\]

Using again that \( \beta \) is a bijection and Proposition 3.2 (1.), we have

\[
(\beta_* \mathcal{O}_X|_{D(f)})_q \cong (\mathcal{O}_X|_{D(f)})_{\beta^{-1}(q)} \cong (\mathcal{O}_X|_{D(f)})_{p} \cong A_p.
\]

Thus we only need to show that \( (A_f)_{\beta^{-1}(q)} \) is isomorphic to \( A_p \). But this is clear using the following maps:

\[
A_p \longrightarrow (A_f)_{\beta^{-1}(q)}
\]

\[
\begin{array}{ccc}
 a & \mapsto & \frac{q}{m} \\
 b & \mapsto & \frac{q}{m}
\end{array}
\]

and

\[
(\beta_* \mathcal{O}_X|_{D(f)})_q \longrightarrow A_p
\]

\[
\begin{array}{ccc}
 \frac{q}{m} & \mapsto & af^m \\
 \frac{b}{m} & \mapsto & \frac{b}{m^n}
\end{array}
\]

\( \square \)

**Exercise.** [2, Chapter II, Exercise 2.2] Let \( (X, \mathcal{O}_X) \) be a scheme, and let \( U \subseteq X \) be an open subset. Show that \( (U, \mathcal{O}_X|_U) \) is a scheme. We call this the *induced scheme structure* on the open set \( U \), and we refer to \( (U, \mathcal{O}_X|_U) \) as an *open subscheme* of \( X \).

**Solution.** Let \( x \in X \), and let \( V \) be an open neighbourhood of \( x \) such that \( (V, \mathcal{O}_V) \) is an affine scheme. As \( U \cap V \) is a non-empty open set in \( V \), there exists \( f \in \mathcal{O}_V(V) \) such that \( D(f) \subseteq U \cap V \) and \( x \in D(f) \). By Proposition 3.2 (2.), \( (D(f), \mathcal{O}_X|_{D(f)}) \) is an affine scheme, and thus \( (U, \mathcal{O}_X|_U) \) is a scheme. \( \square \)

**Exercise.** [2, Chapter II, Exercise 2.4] Let \( A \) be a ring and let \( (X, \mathcal{O}_X) \) be a scheme. Given a morphism \( f : X \to \text{Spec} A \), we have the associated map on sheaves \( f^\ast : \mathcal{O}_{\text{Spec} A} \to f_* \mathcal{O}_X \). Taking global sections we obtain a homomorphism \( A \to \mathcal{O}_X(X) \). Thus there is a natural map

\[
\alpha : \text{Mor}_{\text{Set}}(X, \text{Spec} A) \longrightarrow \text{Mor}_{\text{Ring}}(A, \mathcal{O}_X(X)).
\]

Show that \( \alpha \) is bijective.
**Solution.** We will show that there is a counit-unit adjunction between these two contravariant functors:

\[
\Gamma : \mathcal{S}ch \rightarrow \mathcal{E}Ring
\]

and

\[
\text{Spec} : \mathcal{E}Ring \rightarrow \mathcal{S}ch
\]

We can write

\[
\begin{align*}
\Gamma & : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X), \\
\text{Spec} & : A \mapsto (\text{Spec } A, \mathcal{O}_{\text{Spec } A}).
\end{align*}
\]

The first natural transformation is

\[
\varepsilon : 1_{\mathcal{E}Ring} \rightarrow \Gamma \circ \text{Spec}
\]

\[
\varepsilon_A : A \rightarrow \mathcal{O}_{\text{Spec } A}(\text{Spec } A)
\]

\[
a \mapsto (s_a : p \mapsto a_1, \, \forall p \in \text{Spec } A).
\]

The second natural transformation is

\[
\eta : 1_{\mathcal{S}ch} \rightarrow \text{Spec} \circ \Gamma
\]

\[
\eta_{(X, \mathcal{O}_X)} : (X, \mathcal{O}_X) \mapsto (\text{Spec } \mathcal{O}_X(X), \mathcal{O}_{\text{Spec } \mathcal{O}_X(X)})
\]

\[
\eta_{(X, \mathcal{O}_X)} = (\phi, \phi^\sharp),
\]

where \(\phi\) is the following map:

\[
\phi : X \rightarrow \text{Spec } \mathcal{O}_X(X)
\]

\[
x \mapsto i_x^{-1}(m_x),
\]

where \(m_x\) is the maximal ideal of \(\mathcal{O}_{X,x}\) (which is unique as \((X, \mathcal{O}_X)\) is scheme), and \(i_x\) is the following morphism of rings \(i_x : \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}\). We now show that \(\phi\) is a continuous map (on a basis of open subset of \(\text{Spec } \mathcal{O}_X(X)\)), using the following lemma.

**Lemma 3.1.** Let \(U \subseteq X\), \(U\) is open if and only if there exists an open covering \(\{V_i\}_{i \in I}\) of \(X\) such that \(U \cap V_i\) is an open subset in \(V_i\) for all \(i \in I\).

Let \(f \in \text{Spec } \mathcal{O}_X(X)\),

\[
\phi^{-1}(D(f)) = X_f = \{x \in X : \frac{f}{1} \notin m_x\}.
\]

Covering \(X\) by open affine subschemes, and using Lemma 3.1, we see that \(\phi\) is continuous. We define the morphism of sheaves

\[
\phi^\sharp : \mathcal{O}_{\text{Spec } \mathcal{O}_X(X)} \rightarrow \phi_* \mathcal{O}_X.
\]
By definition of $O_{\text{Spec} \mathcal{O}_X}$ and $\varphi_* \mathcal{O}_X$, it suffices to define $\varphi^\#$ on stalks. We can use the local morphisms $\mathcal{O}_X(x)_{i^{-1}(m_x)} \to \mathcal{O}_{X,x}$ induced by $i_x : \mathcal{O}_X \to \mathcal{O}_{X,x}$.

Using the following isomorphism

$$O_{\text{Spec} \mathcal{O}_X}(\text{Spec } \mathcal{O}_X) \cong \mathcal{O}_X(\text{Spec } \mathcal{O}_X)$$

we can easily compute the first and second counit-unit equations. The first equation

$$\Gamma(\eta_{(X, \mathcal{O}_X)}) \circ \varepsilon_{(X, \mathcal{O}_X)} = 1_{\Gamma(X, \mathcal{O}_X)}$$

can be decomposed into two parts:

$$\varepsilon_{(X, \mathcal{O}_X)} : \mathcal{O}_X \to O_{\text{Spec} \mathcal{O}_X}(\text{Spec } \mathcal{O}_X)$$

$$a \mapsto s_a,$$

and

$$\Gamma(X, \mathcal{O}_X) : O_{\text{Spec} \mathcal{O}_X}(\text{Spec } \mathcal{O}_X) \to \mathcal{O}_X$$

$$s \mapsto \varphi^\#(s).$$

Thus it is clear that $\Gamma\eta \circ \varepsilon = 1_{\Gamma}$. The second equation

$$\text{Spec}(\varepsilon_A) \circ \eta_{\text{Spec}(A)} = 1_{\text{Spec}(A)}$$

can be decomposed into two parts:

$$\eta_{\text{Spec}(A)} : (\text{Spec } A, \mathcal{O}_{\text{Spec} A}) \to \text{Spec}(\mathcal{O}_{\text{Spec} A}(\text{Spec } A))$$

$$p \mapsto i^{-1}(m_p) = p,$$

and we can verify that

$$\text{Spec}(\varepsilon_A) : \text{Spec}(\mathcal{O}_{\text{Spec} A}(\text{Spec } A)) \to (\text{Spec } A, \mathcal{O}_{\text{Spec} A})$$

is the identity. Thus it is clear that $\eta \text{ Spec} \circ \varepsilon = 1_{\text{Spec}}$, and we have the counit-unit adjunction.

Exercise. [2, Chapter II, Exercise 2.5] Describe Spec $\mathbb{Z}$, and show that it is a final object for the category of schemes, i.e., each scheme $X$ admits a unique morphism to Spec $\mathbb{Z}$.

Solution. The only prime ideals of $\mathbb{Z}$ are $(0)$ and the ideals of the form $(p)$ where $p$ is a prime number. As the ideals of $\mathbb{Z}$ are $(n)$ where $n \in \mathbb{N}$, the closed subsets are $V((n)) = \{(p) : p \text{ prime, } p | n \}$.

As $\mathbb{Z}$ is an initial object in the category $\mathfrak{C}\mathfrak{R}ing$, using Exercise 3.2 with $A = \mathbb{Z}$, it is clear that $(\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$ is a final object in the category of schemes, $\mathfrak{S}ch$. □

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Exercise. [2, Chapter II, Exercise 2.10] Describe $\text{Spec } \mathbb{R}[X]$. How does its topological space compare to $\mathbb{R}$? To $\mathbb{C}$?

Solution. The prime ideals of $\mathbb{R}[X]$ are $(0)$, all the ideals generated by $X - a$ where $a \in \mathbb{R}$, and all the ideals generated by $(X - \alpha)(X - \bar{\alpha})$ where $\alpha \in \mathbb{C}$ and $\Im \alpha \neq 0$. Thus the following application:

$$
\mathbb{R} \rightarrow \text{Spec } \mathbb{R}[X] \\
a \mapsto (X - a)
$$

is an injection, while the following application

$$
\mathbb{C} \rightarrow \text{Spec } \mathbb{R}[X] \\
\alpha \mapsto ((X - \alpha)(X - \bar{\alpha})) \quad \text{if } \Im \alpha \neq 0 \\
\alpha \mapsto (X - \alpha) \quad \text{if } \Im \alpha = 0
$$

is a surjection. \qed
References

