SEMESTER PROJECT

Characters and triangle generation of the simple Mathieu group $M_{11}$

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Chapter 1

Introduction

The aim of this project was to learn about the representation and character theory of finite groups over a field of characteristic zero. After reading some references (principally [4]) and doing a couple of exercises, the idea was to apply the theory to a concrete example. In this project, we first construct the character table of the smallest simple Mathieu group $M_{11}$. Recall that the latter belongs to the family of 26 sporadic groups, which contains the finite simple groups which are neither cyclic of prime order, alternating nor of Lie type.

Theorem 1.1

The character table of $M_{11}$ is as follows

<table>
<thead>
<tr>
<th>$g_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8A</th>
<th>8B</th>
<th>11A</th>
<th>11B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>C_{M_{11}}(g_i)</td>
<td>$</td>
<td>7920</td>
<td>48</td>
<td>18</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$i_2$</td>
<td>$-i_2$</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-i_2$</td>
<td>$i_2$</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b_{11}$</td>
<td>$b_{11}$</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$b_{11}$</td>
<td>$b_{11}$</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>44</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_9$</td>
<td>45</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{10}$</td>
<td>55</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1: Character table of $M_{11}$

Here $i_2$ and $b_{11}$ respectively denote $i\sqrt{2}$ and $\frac{-1+i\sqrt{11}}{2}$.

Given a triple $(p_1, p_2, p_3)$ of primes, we then use the character table above in order to determine whether or not $M_{11}$ is a $(p_1, p_2, p_3)$-group, where a group $G$ is called a $(p_1, p_2, p_3)$-group if it is generated by two elements of orders $p_1$ and $p_2$ whose product has order $p_3$. Note that given such a triple of primes, a non-cyclic group is a $(p_1, p_2, p_3)$-group if and only if there is a surjective homomorphism
from the triangle group $T$ to $G$, where

$$T = T_{p_1,p_2,p_3} = (x_1, x_2, x_3 : x_1^{p_1} = x_2^{p_2} = x_3^{p_3} = x_1x_2x_3 = 1).$$

Observe that if

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1,$$

then $T$ is either a solvable group or isomorphic to $A_5$ (see [1, p.361] for more details). Since $|A_5| < |M_{11}|$ and the homomorphic image of a solvable group is itself a solvable group, one can conclude that there is no surjective homomorphism from $T$ to $M_{11}$. We therefore assume hereafter that $(p_1, p_2, p_3)$ is a triple of primes satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1,$$

so that $T$ is a hyperbolic triangle group. We then call $(p_1, p_2, p_3)$ a hyperbolic triple of primes. Geometrically, if $\Delta$ is a hyperbolic triangle having angles of sizes $\frac{\pi}{p_1}$, $\frac{\pi}{p_2}$ and $\frac{\pi}{p_3}$, then $T = T_{p_1,p_2,p_3}$ can be viewed as the group generated by rotations of angles $\frac{\pi}{p_1}$, $\frac{\pi}{p_2}$ and $\frac{\pi}{p_3}$ around the corresponding vertices of $\Delta$ in the hyperbolic plane.

Note that if $\text{lcm}(p_1, p_2, p_3)$ does not divide the order of a finite non-cyclic group $G$, then a homomorphism $\phi$ from $T$ to $G$ sends one of the generators $x_1$, $x_2$ or $x_3$ to 1 and so $\text{Im}(\phi)$ is a cyclic group. Hence $G$ is not a $(p_1,p_2,p_3)$-group.

In our case, as

$$|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11,$$

we only consider the following hyperbolic triples of primes:

1. $(p_1, p_2, p_3) \in \{(2, 3, 11), (2, 5, 11), (3, 5, 11)\}$.
2. $(p_1, p_2, p_3) \in \{(2, 5, 5), (2, 11, 11), (3, 3, 5), (3, 3, 11), (3, 5, 5), (3, 11, 11), (5, 5, 11), (5, 11, 11)\}$.
3. $(p_1, p_2, p_3) \in \{(5, 5, 5), (11, 11, 11)\}$.

We determine the hyperbolic triples $(p_1, p_2, p_3)$ of primes for which $M_{11}$ is a $(p_1,p_2,p_3)$-group.

**Theorem 1.2**

The simple Mathieu group $M_{11}$ is a $(p_1,p_2,p_3)$-group if and only if

1. $(p_1, p_2, p_3) \in \{(2, 5, 11), (3, 5, 11)\}$, or
2. $(p_1, p_2, p_3) \in \{(2, 11, 11), (3, 11, 11), (5, 5, 11), (5, 11, 11)\}$, or
3. $(p_1, p_2, p_3) \in \{(5, 5, 5), (11, 11, 11)\}$.

Finally, given a hyperbolic triple $(p_1, p_2, p_3)$ of primes for which $M_{11}$ is a $(p_1,p_2,p_3)$-group, we determine the probability that a randomly chosen homomorphism from $T = T_{p_1,p_2,p_3}$ to $M_{11}$ is surjective.
Theorem 1.3
Suppose $M_{11}$ is a $(p_1, p_2, p_3)$-group for a given hyperbolic triple $(p_1, p_2, p_3)$ of primes and let $P_{p_1, p_2, p_3}$ be the probability that a randomly chosen homomorphism from $T = T_{p_1, p_2, p_3}$ to $M_{11}$ is surjective. Then the following assertions hold

(i) $P_{2,5,11} = \frac{2|M_{11}|}{6|M_{11}| + 1}$

(ii) $P_{2,11,11} = \frac{2|M_{11}|}{4|M_{11}| + 1441}$

(iii) $P_{3,5,11} = \frac{10|M_{11}|}{18|M_{11}| + 1}$

(iv) $P_{3,11,11} = \frac{14|M_{11}|}{18|M_{11}| + 1441}$

(v) $P_{5,5,5} = \frac{38016}{120701}$

(vi) $P_{5,5,11} = \frac{34|M_{11}|}{54|M_{11}| + 1585}$

(vii) $P_{5,11,11} = \frac{46|M_{11}|}{54|M_{11}| + 1441}$

(viii) $P_{11,11,11} = \frac{30|M_{11}|}{34|M_{11}| + 1441}$

The layout of the project is as follows. In Chapter 2, we prove Theorem 1.1 and we prove Theorems 1.2 and 1.3 in Chapter 3.
Chapter 2

Character table of $M_{11}$

2.1 First observations

The simple Mathieu group $M_{11}$ can be defined using different approaches. One of them is to see $M_{11}$ as the only group of order 7920 acting sharply 4-transitively on a set of 11 elements, with point-stabilizer isomorphic to $A_6 \cdot 2$ (for more details on this construction, see [7]). It is also possible to define $M_{11}$ using Steiner Systems (see also [7]), or finally to visualize it as a subgroup of the largest Mathieu group $M_{24}$ using the theory of Golay Codes (see [3]). However, in order to construct the character table of $M_{11}$, we only use the following facts.

(1) The group $M_{11}$ is simple.

(2) There are ten conjugacy classes in $M_{11}$, with representatives given in the table below.

<table>
<thead>
<tr>
<th>$g_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8A</th>
<th>8B</th>
<th>11A</th>
<th>11B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\langle g_i \rangle</td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$</td>
<td>C_{M_{11}}(g_i)</td>
<td>$</td>
<td>7920</td>
<td>48</td>
<td>18</td>
<td>8</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

(3) $M_{11}$ acts sharply 4-transitively on a set of 11 elements.

(4) $M_{11}$ acts 3-transitively on a set of 12 elements with 3-point stabilizer isomorphic to $S_3$.

2.2 Cycle Shapes

Since there are (at least) two actions of $M_{11}$ on two sets with respective cardinalities 11 and 12, one can deduce that there are two homomorphisms $\phi$ and $\psi$ from $M_{11}$ to $S_{11}$ and $S_{12}$ respectively. Furthermore, since $M_{11}$ is a simple group, those homomorphisms are both injective and then one can visualize $M_{11}$ as a subgroup of $S_{11}$ (via the embedding $\phi$) or of $S_{12}$ (via the embedding $\psi$). The goal of this section is to determine the cycle shapes of the representatives of the conjugacy classes of $M_{11}$ under these two embeddings. We first begin with
an elementary result which shows that $M_{11}$ consists only of even permutations when viewed as a subgroup of $S_{11}$ or $S_{12}$.

**Lemma 2.2.1**

*If $G \leq S_n$ is simple and non-cyclic, then every permutation of $G$ is even.*

**Proof.** Assume for a contradiction that there is an odd permutation in $G$ and consider $N = G \cap A_n$. Since it is clear that $N \triangleleft G$, it suffices to show that $1 \neq N$ and $N \neq G$. However, we supposed the existence of an odd permutation in $G$ and so the latter affirmation is clear. Finally, the fact that $N \neq 1$ just means that there is a non-trivial even permutation in $G$, which is true since $G$ is non-cyclic. \qed 

### 2.2.1 Action of $M_{11}$ on $\{1, \ldots, 11\}$

We first determine the cycle shapes of the representatives of the conjugacy classes in $M_{11}$ embedded in $S_{11}$.

**Proposition 2.2.2**

*The cycle shapes of the representatives of the conjugacy classes in $M_{11}$ seen as a subgroup of $S_{11}$ are as follows.*

<table>
<thead>
<tr>
<th>Order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>11</th>
</tr>
</thead>
</table>

**Proof.**

1) By Lemma 2.2.1, an element of order 2 can only take the following cycle shapes:

   - $(2)(2)$
   - $(2)(2)(2)(2)$

   However, we can rule out the possibility $(2)(2)$ since the action is sharply-4-transitive and so every element of order 2 in $M_{11} \leq S_{11}$ is a product of four transpositions.

2) An element of order 3 can only take the following shapes:

   - $(3)$
   - $(3)(3)$
   - $(3)(3)(3)$

   Again, we can eliminate the possibilities $(3)$ and $(3)(3)$ because of the sharply 4-transitive action and so we are left with every element of order 3 having cycle shape $(3)(3)(3)$. 


3) By Lemma 2.2.1, we see that an element of order 4 can only take the following shapes:

\[(4)(2)\]
\[(4)(2)(2)(2)\]
\[(4)(4)\]

However, taking the square of such an element should give us an element of order 2, which by 2) must be of the shape \((2)(2)(2)(2)\). We conclude that an element of order 4 has cycle shape \((4)(4)\).

We can deal with the remaining cases in a similar way. The details are left to the reader.

\[\square\]

2.2.2 Action of \(M_{11}\) on \(\{1, \ldots, 12\}\)

We now determine the cycle shapes of the representatives of the conjugacy classes in \(M_{11}\) embedded in \(S_{12}\).

**Proposition 2.2.3**

The cycle shapes of the representatives of the conjugacy classes in \(M_{11}\) seen as a subgroup of \(S_{12}\) are as follows.

<table>
<thead>
<tr>
<th>Order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shape</td>
<td>Id</td>
<td>((2)(2)(2))</td>
<td>((3)(3))</td>
<td>((4)(2)(2))</td>
<td>((5)(5))</td>
<td>((6)(3)(2))</td>
<td>((8)(4))</td>
<td>((11))</td>
</tr>
</tbody>
</table>

**Proof.**

1) By Lemma 2.2.1, an element of order 2 can only take the following shapes:

\[(2)(2)\]
\[(2)(2)(2)(2)\]
\[(2)(2)(2)(2)(2)\]

2) An element of order 3 can only take the following shapes:

\[(3)\]
\[(3)(3)\]
\[(3)(3)(3)\]
\[(3)(3)(3)(3)\]

3) By Lemma 2.2.1, an element of order 4 can only take the following shapes:

\[(4)(2)\]
\[(4)(2)(2)(2)\]
\[(4)(4)\]
\[(4)(4)(2)(2)\]
However, we can rule out the possibilities (4)(2) and (4)(4) since the stabilizer of 3 points is isomorphic to $S_3$ and so does not contain an element of order 4. This leaves us with two choices, i.e.

$$(4)(2)(2)(2) \text{ or } (4)(4)(2)(2).$$

4) It is easy to see that an element of order 5 must be of the form (5)(5) since (5) would be an element of order 5 stabilizing 3 points.

5) By Lemma 2.2.1 and using the fact that no element of order 6 can stabilize 3 points, we see that an element of order 6 can only take the following shapes:

$$(6)(6)$$
$$(6)(2)(2)(2)$$
$$(6)(3)(2)$$

However, we can eliminate the first two possibilities since


Hence an element of order 6 is of the form (6)(3)(2). Cubing and squaring give us respectively the shapes of elements of orders 2 and 3. Furthermore, as the square of an element of order 4 is an element of shape (2)(2)(2)(2), we deduce that an element of order 4 has shape (4)(4)(2)(2).

We can deal with the remaining cases in a similar way. The details are left to the reader.

\[ \square \]

### 2.3 Permutation modules and permutation characters

In this subsection, we calculate four non-trivial irreducible characters of $M_{11}$ which come from the two permutation modules associated to the embeddings presented above. Indeed, recall that if $G$ is a subgroup of $S_n$, then the function $\pi : G \to \mathbb{R}$ defined by

$$\pi(g) = |\text{fix}(g)| - 1$$

is a character of $G$. Using the cycle shapes calculated before, we can compute the characters $\pi_1$ and $\pi_2$ associated to the corresponding actions. An easy calculation gives us the following:

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8A</th>
<th>8B</th>
<th>11A</th>
<th>11B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Now recall that if $G$ is a finite group and $\phi, \psi$ are two characters of $G$, then their inner product is defined by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \psi(g).$$
Moreover, the following result is necessary to determine whether a given character of $M_{11}$ is irreducible or not (see [4, p.143] for a proof).

**Proposition 2.3.1**
Assume $G$ is a finite group and let $\chi$ be a character of $G$. Then $\chi$ is irreducible if and only if

$$\langle \chi, \chi \rangle = 1.$$ 

One can now check that the previously computed characters $\pi_1$ and $\pi_3$ are irreducible characters of $M_{11}$, using Proposition 2.3.1.

Before going any further, recall the following key proposition, which can be found in [4, p.198].

**Proposition 2.3.2**
Let $G$ be a finite group and $\chi : G \to \mathbb{C}$ the complex character associated to a $\mathbb{C}G$-module $V$. Define $\chi_S$ to be the character of the $\mathbb{C}G$-module $S^2(V)$ and $\chi_A$ to be the character of the $\mathbb{C}G$-module $\wedge^2(V)$. Then for every $g \in G$, we have

$$\chi_S(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)),$$

$$\chi_A(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

In our case, using Proposition 2.3.1, one can show that $\pi_1A$ and $\pi_2A$ are two new irreducible characters and hence our character table of $M_{11}$ has now five complete rows.

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8A</th>
<th>8B</th>
<th>11A</th>
<th>11B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>45</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>55</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### 2.4 Congruences and degrees

In this section, we get some information concerning the entries of the columns of the character table of $M_{11}$ corresponding to the elements of order 2, 3, 4, 5 and 6. In fact, we first recall two important results which turn out to be very useful in our computations (these theorems and their proofs can be found in [4, pp. 250-251, 256]. In the following two statements, $G$ denotes a finite group.

**Theorem 2.4.1**
Let $g \in G$ be an element of order $n$. Suppose that $g$ is conjugate to $g^k$ for every $1 \leq k \leq n$ such that $(k,n)=1$. Then for all character $\chi$ of $G$, $\chi(g)$ is an integer.
Theorem 2.4.2
Let \( g \in G \) be an element of order \( p^k \) (where \( p \) is a prime number) and \( \chi \) be a character of \( G \) such that \( \chi(g) \) is an integer. Then
\[
\chi(g) \equiv \chi(1) \mod p.
\]

Recall the well-known fact that the number of irreducible characters of a finite group \( G \) is equal to the number of its conjugacy classes. In our case, this implies that there are five more irreducible characters to determine, which we denote by \( \chi_6, \chi_7, \chi_8, \chi_9 \) and \( \chi_{10} \). Using Theorem 2.4.1, we get that every column of the character table of \( M_{11} \) is entirely composed of integers, except possibly the last four. Before going any further, let us recall the following orthogonality relations for \( M_{11} \), which can be deduced from [4, pp. 161-162], for example.

Proposition 2.4.3 (Column orthogonality relations)
For all \( g, h \in M_{11} \), we have
\[
\sum_{i=1}^{10} \chi_i(g)\chi_i(h) = \begin{cases} |C_{M_{11}}(g)| & \text{if } g \text{ and } h \text{ are conjugate}, \\ 0 & \text{otherwise.} \end{cases}
\]

Proposition 2.4.4 (Row orthogonality relations)
For all \( 1 \leq i, j \leq 10 \), we have
\[
\langle \chi_i, \chi_j \rangle = \delta_{ij}.
\]

Now Proposition 2.4.3 applied to the fourth column yields
\[
\chi_i(4) = 0 \forall i \in \{6, 7, 8, 9, 10\}
\]
and hence Theorem 2.4.2 with \( g = 4 \) yields that the degrees of the missing characters \( \chi_i \ (i = 6, \ldots, 10) \) are even. This forces \( \chi_i(2) \) to be even too for each \( i = 6, \ldots, 10 \), by Theorem 2.4.2. We now apply Proposition 2.4.3 to a couple of pairs of representatives to determine the global composition of the columns 2, 3, 4, 5 and 6.

Orth(2,2): We get \( 24 = \chi_6(2)^2 + \chi_7(2)^2 + \chi_8(2)^2 + \chi_9(2)^2 + \chi_{10}(2)^2 \). The possibilities available for \( \chi_i(2) \ (i = 6, \cdots, 10) \) are \((\pm4, \pm2, \pm2, 0, 0)\) and \((\pm3, \pm3, \pm2, \pm1, \pm1)\), in some order. However, since the entries of the second column are all supposed to be even, we can rule out the last possibility and so we know that this column is of the form
\[
(1, 2, -3, 3, -1, \pm4, \pm2, \pm2, 0, 0),
\]
the last five entries in some order.

Orth(3,3): We get \( 11 = \chi_6(3)^2 + \chi_7(3)^2 + \chi_8(3)^2 + \chi_9(3)^2 + \chi_{10}(3)^2 \), which gives us two possibilities for the entries of the third column (again in some order) :
\[
(1, 1, 0, 2, 1, \pm3, \pm1, \pm1, 0, 0) \text{ or } (1, 1, 0, 2, 1, \pm2, \pm2, \pm1, \pm1).
\]
**Orth**(5, 5): We get the possibility \((1, 0, 0, 1, 0, \pm 1, \pm 1, 0, 0)\), the last five entries in some order.

**Orth**(6, 6): We get the possibility \((1, -1, 0, 1, \pm 1, \pm 1, 0, 0)\), the last five entries in some order.

Assume that the second column is \((1, 2, -3, 3, -1, \pm 4, \pm 2, 0, 0)\) (we shall determine the signs later). Then applying wisely Proposition 2.4.3, we get the following partial table.

<table>
<thead>
<tr>
<th>Element</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_6)</td>
<td>(\pm 4)</td>
<td>(\mp 1)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>(\pm 1)</td>
</tr>
<tr>
<td>(\chi_7)</td>
<td>(\pm 2)</td>
<td>(\mp 1)</td>
<td>0</td>
<td>0</td>
<td>(\mp 1)</td>
</tr>
<tr>
<td>(\chi_8)</td>
<td>(\pm 2)</td>
<td>(\mp 1)</td>
<td>0</td>
<td>0</td>
<td>(\mp 1)</td>
</tr>
<tr>
<td>(\chi_9)</td>
<td>0</td>
<td>(\pm 2)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>0</td>
</tr>
<tr>
<td>(\chi_{10})</td>
<td>0</td>
<td>(\pm 2)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that we now only need to determine one entry in each row, since they are all related to one another. Furthermore, if we take a better look at the characters \(\chi_7\) and \(\chi_8\), we can use Theorem 2.4.2 applied to an element of order 5 and deduce that their degrees must be even and divisible by 5. Using these facts together with Proposition 2.4.3 applied to the first column, we can deduce that

\[\chi_7(1), \chi_8(1) \in \{10, 20, 30, 40, 50\}.
\]

Then we have to look at every possible situation and rule out systematically every single one which leads us to a contradiction. Although it is not complicated (it only uses orthogonality relations and the fact that every entry of columns one to six has to be an integer), it is a very lengthy task, so we claim that the only possibility is

\[\chi_7(1) = \chi_8(1) = 10\]

and leave the details to the reader.

Now using Theorem 2.4.2 yields

\[\chi_7(3) = \chi_8(3) = 1\]

and hence the table can be revised as follows.

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_6)</td>
<td>-</td>
<td>(\pm 4)</td>
<td>(\mp 1)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>(\pm 1)</td>
</tr>
<tr>
<td>(\chi_7)</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_8)</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_9)</td>
<td>-</td>
<td>0</td>
<td>(\pm 2)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>0</td>
</tr>
<tr>
<td>(\chi_{10})</td>
<td>-</td>
<td>0</td>
<td>(\pm 2)</td>
<td>0</td>
<td>(\mp 1)</td>
<td>0</td>
</tr>
</tbody>
</table>
Using the same method to determine the remaining degrees and then congruences to compute the signs, we finally get the following table.

<table>
<thead>
<tr>
<th>Element</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8A</th>
<th>8B</th>
<th>11A</th>
<th>11B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>10</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>45</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>55</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>44</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>10</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_9$</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_{10}$</td>
<td>16</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

### 2.5 Complex characters and conclusion

One can now use Proposition 2.4.4 to first show that

$$\chi_6(8A) = \chi_6(8B) = \chi_6(11A) = \chi_6(11B) = 0.$$  

Finally, we use appropriately Propositions 2.4.3 and 2.4.4 to determine the last 16 entries of the character table of $M_{11}$, keeping in mind that if $\chi$ is an irreducible character of a group $G$, then so is $\overline{\chi}$, where $\overline{\chi}: G \to \mathbb{C}$ is defined by

$$\overline{\chi}(g) = \chi(g), \quad g \in G.$$  

This completes the proof of Theorem 1.1.
Chapter 3

Triangle groups and $M_{11}$

3.1 Preliminaries

Given a hyperbolic triangle group $T = T_{p_1, p_2, p_3}$ arising from a hyperbolic triple $(p_1, p_2, p_3)$ of primes, we use the character theory of $M_{11}$ to calculate the size of $\text{Hom}(T, M_{11})$ and we then use the subgroup structure of $M_{11}$ together with the character theory of its maximal subgroups to determine the size of

$$\bigcup_{M \in \mathfrak{M}} \text{Hom}(T, M),$$

where $\mathfrak{M}$ denotes the set of all maximal subgroups of $M_{11}$.

In this way, we are able to compute the number of surjective homomorphisms from $T$ to $M_{11}$, which is given by

$$|\text{Hom}(T, M_{11})| - \left| \bigcup_{M \in \mathfrak{M}} \text{Hom}(T, M) \right|$$

and determine whether or not $M_{11}$ is a $(p_1, p_2, p_3)$-group (indeed, $M_{11}$ is a $(p_1, p_2, p_3)$-group if and only if the latter quantity is positive).

Given a hyperbolic triple $(p_1, p_2, p_3)$ of primes for which $M_{11}$ is a $(p_1, p_2, p_3)$-group, we then determine the probability that a randomly chosen homomorphism from $T$ to $M_{11}$ is surjective, which is given by

$$1 - |\text{Hom}(T, M_{11})|^{-1} \left| \bigcup_{M \in \mathfrak{M}} \text{Hom}(T, M) \right|.$$

Let us now fix some notation which will be used in the remainder of the project. For any group $G$, we set

$$\text{Hom}^*(T, G) = \{ \phi \in \text{Hom}(T, G) : \phi(x_1) \neq 1, \phi(x_2) \neq 1, \phi(x_3) \neq 1 \}$$

and

$$\text{Hom}^1(T, G) = \{ \phi \in \text{Hom}(T, G) : \phi \text{ is surjective} \}.$$

Moreover, given a hyperbolic triangle group $T = T_{p_1, p_2, p_3}$, we let $P = P_{p_1, p_2, p_3}$ be the probability that a randomly chosen homomorphism from $T$ to $M_{11}$ is surjective.
Finally, given a finite group $G$ and a triple $C = (C_1, C_2, C_3)$ of conjugacy classes $C_i$ of $G$, we define

$$\text{Hom}_C(T,G) = \{ \phi \in \text{Hom}(T,G) : \phi(x_i) \in C_i \}.$$ 

**Proposition 3.1.1**

The following assertions hold.

(i) 

$$|\text{Hom}_C(T,G)| = |C_1||C_2||C_3| \sum_{\chi} \frac{\chi(g_1)\chi(g_2)\chi(g_3)}{|G|},$$

where $\chi$ runs over the set of irreducible characters of $G$ and $g_i$ is a representative of the conjugacy class $C_i$.

(ii) 

$$|\text{Hom}(T,G)| = \sum_C |\text{Hom}_C(T,G)|,$$

where the sum is over all $C = (C_1, C_2, C_3)$ such that each $C_i$ is a conjugacy class of $G$ of elements of order dividing $p_i$.

(iii) 

$$|\text{Hom}^*(T,G)| = \sum_C |\text{Hom}_C(T,G)|,$$

where the sum is over all $C = (C_1, C_2, C_3)$ such that each $C_i$ is a conjugacy class of $G$ of elements of order dividing $p_i$.

**Proof.** Parts (ii) and (iii) follow directly from the definition of $\text{Hom}_C(T,G)$. For a proof of part (i), see [6, Proposition 3.2]. \qed

**Lemma 3.1.2**

Let $G$ be a finite group. If $G$ is non-cyclic, then

(i) 

$$\text{Hom}^1(T,G) = \text{Hom}^*(T,G) \setminus \bigcup_{M \in \mathbb{N}} \text{Hom}^*(T,M).$$

(ii) 

$$P = \frac{|\text{Hom}^1(T,M_{11})|}{|\text{Hom}(T,M_{11})|}$$

**Proof.** Part (ii) follows directly from part (i). To prove this latter, it suffices to note that 

$$\text{Hom}^1(T,G) \subseteq \text{Hom}^*(T,G).$$ \qed
**Lemma 3.1.3**

Let $G$ be a non-cyclic finite group.

**(i)** If $\text{lcm}(p_1, p_2, p_3)$ does not divide $|G|$, then $G$ is not a $(p_1, p_2, p_3)$-group.

**(ii)** If $N \triangleleft G$ and at least two of $p_1, p_2, p_3$ are coprime with $|G : N|$, then

$$\text{Hom}(T, G) = \text{Hom}(T, N).$$

**(iii)** If $L < G$, then

$$\left| \bigcup_{g \in G} \text{Hom}^1(T, L^g) \right| = |G : N_G(L)||\text{Hom}^1(T, L)|.$$

**Proof.** Part (i) has already been proved in the introduction. Let then $N \triangleleft G$ be a normal subgroup of $G$ and $(p_1, p_2, p_3)$ a triple of primes such that at least two of them are coprime with $|G : N|$. First note that we only have to prove that $\text{Hom}(T, G) \subset \text{Hom}(T, N)$, so let $\phi \in \text{Hom}(T, G)$ and consider the quotient map

$$\pi : G \longrightarrow G/N.$$ Clearly, $\phi \in \text{Hom}(T, N)$ if and only if the homomorphism

$$\pi \circ \phi : T \longrightarrow G/N$$

is trivial. This is obviously the case since we supposed that at least two of $p_1, p_2, p_3$ are coprime with $|G : N|$ and then (ii) is proved. Finally, in order to prove part (iii), consider $L < G$ and note that if $g \in G$ is such that $L^g \neq L$, then

$$\text{Hom}^1(T, L) \cap \text{Hom}^1(T, L^g) = \emptyset$$

and hence

$$\left| \bigcup_{g \in G} \text{Hom}^1(T, L^g) \right| = |G : N_G(L)||\text{Hom}^1(T, L)|.$$

In order to apply Lemma 3.1.2 (i) and determine whether $M_{11}$ is a $(p_1, p_2, p_3)$-group, we use the subgroup structure of $M_{11}$ which is given in [2]. We reproduce it here for convenience.

**Proposition 3.1.4**

Any maximal subgroup of $M_{11}$ is conjugate in $M_{11}$ to one of the following

- $A_6 \cdot 2$
- $L_2(11)$
- $3^2 : Q_8 \cdot 2$
- $S_5$
- $2 \cdot S_4$
We end this section by describing how the alternating group $A_5$ sits inside the group $M_{11}$.

**Lemma 3.1.5**

There are two conjugacy classes of subgroups of $M_{11}$ isomorphic to $A_5$, with normalizer $S_5$ and $A_5$ respectively.

### 3.2 Three distinct primes

Assume that $p_1$, $p_2$ and $p_3$ are distinct primes with $p_1 < p_2 < p_3$ and let $T = T_{p_1,p_2,p_3}$. In this situation, observe that

$$\text{Hom}^*(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \phi \text{ is non trivial} \}$$

and so

$$|\text{Hom}(T, M_{11})| = |\text{Hom}^*(T, M_{11})| + 1. \quad (3.1)$$

#### 3.2.1 Total number of homomorphisms

We first determine the size of $\text{Hom}(T, M_{11})$.

**Proposition 3.2.1**

The following assertions hold

(i) \[ |\text{Hom}(T_{2,3,11}, M_{11})| = 2|M_{11}| + 1 \]

(ii) \[ |\text{Hom}(T_{2,5,11}, M_{11})| = 6|M_{11}| + 1 \]

(iii) \[ |\text{Hom}(T_{3,5,11}, M_{11})| = 18|M_{11}| + 1 \]

We first consider part (i). Then $C_1 = 2^{M_{11}}$, $C_2 = 3^{M_{11}}$ and $C_3 = 11^{A_{M_{11}}}$ or $11^{B_{M_{11}}}$. Therefore by Proposition 3.1.1(iii),

$$|\text{Hom}^*(T, M_{11})| = |\text{Hom}(2^{M_{11},3^{M_{11}},11^{A_{M_{11}}}}(T, M_{11}))| + |\text{Hom}(2^{M_{11},3^{M_{11}},11^{B_{M_{11}}}}(T, M_{11}))|.$$

Now by Proposition 3.1.1(i) and the character table of $M_{11}$ given in Theorem 1.1,

\[
|\text{Hom}(2^{M_{11},3^{M_{11}},11^{A_{M_{11}}}}(T, M_{11}))| = \frac{2^{M_{11}}3^{M_{11}}11^{A_{M_{11}}}}{|M_{11}|} \sum_{i=1}^{10} \frac{\chi_i(2)\chi_i(3)\chi_i(11A)}{\chi_i(1)}
\]

\[
= \frac{|M_{11}|^2}{48 \cdot 18 \cdot 11} \left(1 + \frac{1}{5}\right)
\]

\[
= \frac{3 \cdot 2|M_{11}|^2}{25 \cdot 3^2 \cdot 5 \cdot 11}
\]

\[
= |M_{11}|.
\]
CHAPTER 3. TRIANGLE GROUPS AND $M_{11}$

Similarly,

$$|\text{Hom}(T, M_{11})| = |M_{11}|.$$  

Using (3.1), we conclude that

$$|\text{Hom}(T_{2,3,11}, M_{11})| = 2|M_{11}| + 1.$$  

Parts (ii) and (iii) can be dealt in a similar way, completing the proof of Proposition 3.2.1.

3.2.2 Surjective homomorphisms

We now determine the size of $\text{Hom}^1(T, M_{11})$.

**Proposition 3.2.2**

The following assertions hold

(i)  

$$|\text{Hom}^1(T_{2,3,11}, M_{11})| = 0$$

(ii)  

$$|\text{Hom}^1(T_{2,5,11}, M_{11})| = 2|M_{11}|$$

(iii)  

$$|\text{Hom}^1(T_{3,5,11}, M_{11})| = 10|M_{11}|$$

By Lemma 3.1.3(i) and Proposition 3.1.4,

$$\bigcup_{M \in \mathbb{M}} \text{Hom}^*(T, M) = \bigcup_{g \in M_{11}} \text{Hom}^*(T, L_2(11)^g).$$

Moreover, by [5, Proposition 6.1 (i)]

$$\text{Hom}^*(T, L_2(11)) = \text{Hom}^1(T, L_2(11)),$$

i.e. every non-trivial homomorphism from $T$ to $L_2(11)$ is surjective.

Now by Lemma 3.1.3 (iii),

$$\left| \bigcup_{g \in M_{11}} \text{Hom}^*(T, L_2(11)^g) \right| = |M_{11} : N_{M_{11}}(L_2(11))| |\text{Hom}^1(T, L_2(11))|$$

and hence using Lemma 3.1.2(i), we get

$$|\text{Hom}^1(T, M_{11})| = |\text{Hom}^*(T, M_{11})| - |M_{11} : L_2(11)| |\text{Hom}^1(T, L_2(11))|. \quad (3.2)$$

It remains to determine the size of $\text{Hom}^1(T, L_2(11))$. 

Proposition 3.2.3
The following assertions hold

(i) \[ |\text{Hom}^1(T_{2,3,11}, L_2(11))| = 2|L_2(11)| \]

(ii) \[ |\text{Hom}^1(T_{2,5,11}, L_2(11))| = 4|L_2(11)| \]

(iii) \[ |\text{Hom}^1(T_{3,5,11}, L_2(11))| = 8|L_2(11)| \]

Proof. This follows directly from [5, Proposition 6.1 (i)].

Hence, using (3.2), the sizes of \( \text{Hom}^*(T, M_{11}) \) calculated in the previous subsection and the fact that \(|M_{11} : L_2(11)| = 12\), Proposition 3.2.2 is now established.

3.2.3 Conclusion

We finally determine the probability that a randomly chosen homomorphism in \( \text{Hom}(T, M_{11}) \) is surjective.

Theorem 3.2.4
The following assertions hold

(i) \[ P_{2,3,11} = 0 \]

(ii) \[ P_{2,5,11} = \frac{2|M_{11}|}{6|M_{11}| + 1} \]

(iii) \[ P_{3,5,11} = \frac{10|M_{11}|}{18|M_{11}| + 1} \]

Proof. By Lemma 3.1.2(ii), we have

\[ P = \frac{\text{Hom}^1(T, M_{11})}{\text{Hom}(T, M_{11})} \]

and so the results follow from Propositions 3.2.1 and 3.2.2.

3.3 Two equal primes

Assume that exactly two of \( p_1, p_2, p_3 \) are equal, say \( p_1 = p_3 \), and let \( T = T_{p_1,p_1,p_2} \). In this situation, observe that

\[ \text{Hom}^*(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \text{Im}(\phi) \text{ is not a subgroup of } Z_{p_1} \} \]

and so

\[ |\text{Hom}(T, M_{11})| = |\text{Hom}^*(T, M_{11})| + 1 + |\{\text{elements of order } p_1 \text{ in } M_{11}\}|. \quad (3.3) \]
3.3.1 Total number of homomorphisms

We first determine the size of $\text{Hom}(T, M_{11})$.

**Proposition 3.3.1**

The following assertions hold

(i) $|\text{Hom}(T_{2,5,5}, M_{11})| = 9|M_{11}| + 1585$

(ii) $|\text{Hom}(T_{2,11,11}, M_{11})| = 4|M_{11}| + 1441$

(iii) $|\text{Hom}(T_{3,3,5}, M_{11})| = 9|M_{11}| + 441$

(iv) $|\text{Hom}(T_{3,3,11}, M_{11})| = 2|M_{11}| + 441$

(v) $|\text{Hom}(T_{3,5,5}, M_{11})| = 16|M_{11}| + 1585$

(vi) $|\text{Hom}(T_{3,11,11}, M_{11})| = 18|M_{11}| + 1441$

(vii) $|\text{Hom}(T_{5,3,5}, M_{11})| = 9|M_{11}| + 441$

(viii) $|\text{Hom}(T_{5,3,11}, M_{11})| = 2|M_{11}| + 441$

(ix) $|\text{Hom}(T_{5,5,5}, M_{11})| = 16|M_{11}| + 1585$

(x) $|\text{Hom}(T_{5,5,11}, M_{11})| = 18|M_{11}| + 1441$

Again, all the cases are similar and so we treat only case (ii), where $(p_1, p_2, p_3) = (2, 11, 11)$, leaving the others to the reader. Then $C_1 = 2M_{11}$, $C_2 = 11A^{M_{11}}$ or $11B^{M_{11}}$ and $C_3 = 11A^{M_{11}}$ or $11B^{M_{11}}$. By Proposition 3.1.1(iii),

$$|\text{Hom}^*(T, M_{11})| = |\text{Hom}(2M_{11}, 11A^{M_{11}}(T, M_{11}))| + |\text{Hom}(2M_{11}, 11B^{M_{11}}(T, M_{11}))| + |\text{Hom}(2M_{11}, 11A^{M_{11}}(T, M_{11}))| + |\text{Hom}(2M_{11}, 11B^{M_{11}}(T, M_{11}))|.$$  

Now Proposition 3.1.1(i) and the character table of $M_{11}$ given in Theorem 1.1 yield

$$|\text{Hom}(2M_{11}, 11A^{M_{11}}(T, M_{11}))| = \frac{|2M_{11}|^2|11A^{M_{11}}|}{|M_{11}|} \cdot \sum_{i=1}^{10} \chi_i(2)^2\chi_i(11A)\chi_2(11A)$$

$$= \frac{|M_{11}|^2}{2^4 \cdot 3 \cdot 11^2} \left( \frac{33}{15} \right)$$

$$= |M_{11}|.$$
Similarly,
\[
|\text{Hom}_{2^{2M_{11}},11B^{2M_{11}},11A^{M_{11}}}(T, M_{11})| = |\text{Hom}_{2^{2M_{11}},11A^{M_{11}},11B^{M_{11}}}(T, M_{11})| = |\text{Hom}_{(2,11B^{M_{11}},11B^{M_{11}})}(T, M_{11})| = |M_{11}|.
\]

Hence using (3.3) yields
\[
|\text{Hom}(T_{2,11,11}, M_{11})| = 4|M_{11}| + 1 + 1440.
\]
The other cases can be dealt in a similar way, completing the proof of Proposition 3.3.1.

### 3.3.2 Surjective homomorphisms

We now determine the size of $\text{Hom}^1(T, M_{11})$.

**Proposition 3.3.2**

The following assertions hold

(i) $|\text{Hom}^1(T_{2,5,5}, M_{11})| = 0$

(ii) $|\text{Hom}^1(T_{2,11,11}, M_{11})| = 2|M_{11}|$

(iii) $|\text{Hom}^1(T_{3,3,5}, M_{11})| = 0$

(iv) $|\text{Hom}^1(T_{3,3,11}, M_{11})| = 0$

(v) $|\text{Hom}^1(T_{3,5,5}, M_{11})| = 0$

(vi) $|\text{Hom}^1(T_{3,11,11}, M_{11})| = 14|M_{11}|$

(vii) $|\text{Hom}^1(T_{5,5,11}, M_{11})| = 34|M_{11}|$

(viii) $|\text{Hom}^1(T_{5,11,11}, M_{11})| = 46|M_{11}|$

We first consider the case where

$\langle p_1, p_2, p_3 \rangle \in \{(2,11,11), (3,3,11), (3,11,11), (5,5,11), (5,11,11)\}$.

By Lemma 3.1.3(i) and Proposition 3.1.4,
\[
\bigcup_{M \in \mathfrak{M}} \text{Hom}^*(T, M) = \bigcup_{g \in M_{11}} \text{Hom}^*(T, L_2(11)^g).
\]
Moreover, in the cases where
\[(p_1, p_2, p_3) \in \{(2, 11, 11), (3, 11, 11), (5, 11, 11)\},\]
then [5, Proposition 6.1 (ii)] gives
\[
|\text{Hom}^\ast(T, L_2(11))| = |\text{Hom}^1(T, L_2(11))|.
\]

Hence, by Lemma 3.1.3 (iii),
\[
\left| \bigcup_{M \in \mathcal{M}} \text{Hom}^\ast(T, M) \right| = |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))|. \tag{3.4}
\]

Consider now the case where
\[(p_1, p_2, p_3) \in \{(3, 3, 11), (5, 5, 11)\}.
\]

If \(\phi \in \text{Hom}^\ast(T, L_2(11))\), then either \(\phi\) is surjective or has its image included in a maximal subgroup of \(L_2(11)\). One can check in [2] that the maximal subgroups of \(L_2(11)\) which contain an element of order 11 are the Borel subgroups, which are of the form \(J = 11 : 5\).

- In the (3, 3, 11) case, Lemma 3.1.3 (ii) yields
  \[
  \text{Hom}^\ast(T, J) = \text{Hom}^\ast(T, Z_{11}) = \emptyset,
  \]
  which implies that
  \[
  \left| \bigcup_{g \in M_{11}} \text{Hom}^\ast(T, L_2(11)^g) \right| = |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))|,
  \]
  by Lemma 3.1.3 (iii).

- In the (5, 5, 11) case, observe that any non-trivial homomorphism from \(T\) to \(J\) is either surjective or has its image isomorphic to a cyclic group of order 5 or 11. Hence by Lemma 3.1.2 (i),
  \[
  \text{Hom}^\ast(T, J) = \text{Hom}^1(T, J),
  \]
  which gives
  \[
  \left| \bigcup_{g \in M_{11}} \text{Hom}^\ast(T, L_2(11)^g) \right| = |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))| + |M_{11} : J||\text{Hom}^1(T, J)|. \tag{3.5}
  \]

By Lemma 3.1.2 (i), it remains to determine \(|\text{Hom}^1(T, L_2(11))|\) and \(|\text{Hom}^1(T, J)|\), where the latter is only needed for \((p_1, p_2, p_3) = (5, 5, 11)\).
Proposition 3.3.3
The following assertions hold

(i) \[|\text{Hom}^*(T_{2,11,11}, L_2(11))| = 2|L_2(11)|\]

(ii) \[|\text{Hom}^*(T_{3,3,11}, L_2(11))| = 2|L_2(11)|\]

(iii) \[|\text{Hom}^*(T_{3,11,11}, L_2(11))| = 4|L_2(11)|\]

(iv) \[|\text{Hom}^*(T_{5,11,11}, L_2(11))| = 8|L_2(11)|\]

Proof. This follows directly from [5, Proposition 4.1 (v)].

Therefore parts (ii), (iv), (vi) and (viii) of Proposition 3.3.2 follow easily. If \((p_1, p_2, p_3) = (5, 5, 11)\), then Lemma 3.1.2(i) and (3.5) give

\[|\text{Hom}^1(T, M_{11})| = |\text{Hom}^*(T, M_{11})| - |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))| - |M_{11} : J||\text{Hom}^1(T, J)|.\]

Now [5, Proposition 6.1 (ii)] and [5, Proposition 4.1 (ii)] respectively give

\[|\text{Hom}^1(T, L_2(11))| = 12|L_2(11)|\text{ and }|\text{Hom}^1(T, J)| = 440.\]

Hence part (vii) of Proposition 3.3.2 is also established.

Now consider the case where

\((p_1, p_2, p_3) \in \{(2, 5, 5), (3, 3, 5), (3, 5, 5)\}\).

By Proposition 3.1.4, the only maximal subgroups of \(M_{11}\) which contain elements of order 5 are \(A_6 \cdot 2, L_2(11)\) and \(S_5\). Now observe that

\[\text{Hom}^*(T, S_5) = \text{Hom}^*(T, A_5)\text{ and }\text{Hom}^*(T, A_6) = \text{Hom}^1(T, A_5),\]

where the former equality follows from Lemma 3.1.3 (ii) and the latter equality follows from the fact that every maximal subgroup of \(A_5\) is isomorphic to \(A_4, D_{10}\) or \(S_3\) (see [2]). Similarly, by Lemma 3.1.3 (ii),

\[\text{Hom}^*(T, A_6 \cdot 2) = \text{Hom}^*(T, A_6).\]

Using Lemmas 3.1.2(i), 3.1.3(iii) and 3.1.5, one gets the following result.

Lemma 3.3.4

\[|\text{Hom}^1(T, M_{11})| = |\text{Hom}^*(T, M_{11})| - |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))| - |M_{11} : A_6 \cdot 2||\text{Hom}^1(T, A_6)| - |M_{11} : S_5||\text{Hom}^1(T, A_5)| - |M_{11} : A_5||\text{Hom}^1(T, A_5)|.\]
Therefore, we need to compute the size of $\text{Hom}^1(T, L_2(11))$, $\text{Hom}^1(T, A_6)$ and $\text{Hom}^1(T, A_5)$ in the three different cases in which we are interested.

**Proposition 3.3.5**  
The following assertions hold

(i) $|\text{Hom}^1(T_{2,5,5}, L_2(11))| = 4|L_2(11)|$

(ii) $|\text{Hom}^1(T_{3,3,5}, L_2(11))| = 4|L_2(11)|$

(iii) $|\text{Hom}^1(T_{3,5,5}, L_2(11))| = 8|L_2(11)|$

*Proof.* This follows directly from [5, Proposition 6.1 (ii)].  

**Proposition 3.3.6**  
The following assertions hold

(i) $|\text{Hom}^1(T_{2,5,5}, A_6)| = 4|A_6|$

(ii) $|\text{Hom}^1(T_{3,3,5}, A_6)| = 4|A_6|$

(iii) $|\text{Hom}^1(T_{3,5,5}, A_6)| = 4|A_6|$

*Proof.* Since $A_6 \cong L_2(9)$, one can conclude using [5, Proposition 6.1 (ii)].  

**Proposition 3.3.7**  
The following assertions hold

(i) $|\text{Hom}^1(T_{2,5,5}, A_5)| = 2|A_5|$

(ii) $|\text{Hom}^1(T_{3,3,5}, A_5)| = 2|A_5|$

(iii) $|\text{Hom}^1(T_{3,5,5}, A_5)| = 4|A_5|$

*Proof.* Since $A_5 \cong L_2(5)$, one can conclude using [5, Proposition 6.1 (ii)] together with [5, Proposition 4.1 (iv)].  

Now using Lemma 3.3.4 and the sizes of $\text{Hom}^*(T, M_{11})$ calculated in the previous subsection, we get parts (i), (iii) and (v) of Proposition 3.3.2, which were the only ones left to prove.
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3.3.3 Conclusion
We finally determine the probability that a randomly chosen homomorphism in Hom$(T, M_{11})$ is surjective.

**Theorem 3.3.8**
The following assertions hold

(i) $P_{2,5,5} = 0$

(ii) $P_{2,11,11} = \frac{2 |M_{11}|}{4 |M_{11}| + 1441}$

(iii) $P_{3,3,5} = 0$

(iv) $P_{3,3,11} = 0$

(v) $P_{3,5,5} = 0$

(vi) $P_{3,11,11} = \frac{14 |M_{11}|}{18 |M_{11}| + 1441}$

(vii) $P_{5,5,11} = \frac{34 |M_{11}|}{54 |M_{11}| + 1585}$

(viii) $P_{5,11,11} = \frac{46 |M_{11}|}{54 |M_{11}| + 1441}$

**Proof.** By Lemma 3.1.2(ii) we have

$$P = \frac{|\text{Hom}^1(T, M_{11})|}{|\text{Hom}(T, M_{11})|}$$

and so the results follow from Propositions 3.3.1 and 3.3.2. \qed

3.4 Equal primes
Suppose that $p_1 = p_2 = p_3$ and let $T = T_{p_1,p_2,p_3}$. In order to determine the size of Hom$(T, M_{11})$, one must take into account the homomorphisms in

$$S = \text{Hom}(T, M_{11}) \setminus \text{Hom}^+(T, M_{11}).$$

Now

$$S = \text{Hom}^1(T, M_{11}) \cup \text{Hom}^+(T, M_{11}) \cup \text{Hom}^{++}(T, M_{11}) \cup \text{Hom}^{+++}(T, M_{11}),$$

where
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1) $\text{Hom}^I(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \phi \text{ is the trivial homomorphism} \}$,
2) $\text{Hom}^+(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \phi(x_1) = 1, \phi(x_2) \neq 1 \}$,
3) $\text{Hom}^{++}(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \phi(x_2) = 1, \phi(x_1) \neq 1 \}$,
4) $\text{Hom}^{+++}(T, M_{11}) = \{ \phi \in \text{Hom}(T, M_{11}) : \phi(x) = \phi(x_2)^{-1}, \phi(x_1) \neq 1 \}$.

Hence

$$|\text{Hom}(T, M_{11})| = |\text{Hom}^+(T, M_{11})| + 1 + 3 \cdot |\{\text{Elements of order } p_1\}|. \quad (3.6)$$

Given a group $G$, we introduce the following notation:

$$\text{Hom}^{**}(T, G) = \{ \phi \in \text{Hom}^*(T, G) : \text{Im}(\phi) \text{ is not cyclic} \}.$$ 

Since $\text{Hom}^1(T, M_{11}) \subset \text{Hom}^{**}(T, M_{11})$, we have

$$\text{Hom}^1(T, G) = \text{Hom}^{**}(T, G) - \bigcup_{M \in \mathbb{N}} \text{Hom}^{**}(T, M). \quad (3.7)$$

Moreover, note that the set of homomorphisms in $\text{Hom}^*(T, M_{11})$ having cyclic image is given by

$$\text{Hom}^{**}(T, M_{11}) = \{ \phi \in \text{Hom}^*(T, M_{11}) : \phi(x_2) = \phi(x_1)^k, \ 1 \leq k < p_1 - 1 \}$$

and so

$$|\text{Hom}^{**}(T, M_{11})| = |\text{Hom}^*(T, M_{11})| - |\{\text{elements of order } p_1\}|(p_1 - 2). \quad (3.8)$$

### 3.4.1 Total number of homomorphisms

We first determine the size of $\text{Hom}(T, M_{11})$.

**Proposition 3.4.1**

The following assertions hold.

(i)

$$|\text{Hom}(T_{5,5,5}, M_{11})| = 603505$$

(ii)

$$|\text{Hom}(T_{11,11,11}, M_{11})| = 270721$$

First consider the case where $(p_1, p_1, p_1) = (5, 5, 5)$. Then Proposition 3.1.1 together with the character table of $M_{11}$ given in Theorem 1.1 yield:

$$|\text{Hom}^*(T, M_{11})| = \frac{|M_{11}|^2}{5^4} \left( 1 + \frac{1}{11} + \frac{1}{8} - \frac{1}{44} \right) = \frac{378}{5} |M_{11}|.$$
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Similarly, in the case where $(p_1, p_1, p_1) = (11, 11, 11)$, one gets

$$|\text{Hom}^*(T, M_{11})| = |\text{Hom}_{(C_{11A}, C_{11A}, C_{11A})}(T, M_{11})| + |\text{Hom}_{(C_{11A}, C_{11A}, C_{11B})}(T, M_{11})| + |\text{Hom}_{(C_{11A}, C_{11B}, C_{11A})}(T, M_{11})| + |\text{Hom}_{(C_{11B}, C_{11A}, C_{11A})}(T, M_{11})| + |\text{Hom}_{(C_{11B}, C_{11B}, C_{11A})}(T, M_{11})| + |\text{Hom}_{(C_{11B}, C_{11A}, C_{11B})}(T, M_{11})| + |\text{Hom}_{(C_{11B}, C_{11B}, C_{11B})}(T, M_{11})| + |\text{Hom}_{(C_{11B}, C_{11B}, C_{11B})}(T, M_{11})| = 370 \frac{1}{11}|M_{11}|.$$

We then get Proposition 3.4.1 using (3.6).

3.4.2 Surjective homomorphisms

We now determine the size of $\text{Hom}^1(T, M_{11})$.

**Proposition 3.4.2**

The following assertions hold

(i) \[|\text{Hom}^1(T_{5,5,5}, M_{11})| = 24|M_{11}|\]

(ii) \[|\text{Hom}^1(T_{11,11,11}, M_{11})| = 30|M_{11}|\]

We first treat the case where $(p_1, p_2, p_3) = (11, 11, 11)$. By Lemma 3.1.3(i) and Proposition 3.1.4,

$$\bigcup_{M \in \mathbb{N}} \text{Hom}^{**}(T, M) = \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, L_2(11)^g).$$

Moreover, we claim that

$$\text{Hom}^{**}(T, L_2(11)) = \text{Hom}^1(T, L_2(11)).$$

Indeed, if $\phi \in \text{Hom}^{**}(T, L_2(11))$ is not surjective, then $\text{Im}(\phi) \leq J$, where $J = 11:5$ is a Borel subgroup of $L_2(11)$ (see [2]). However, since $J$ only contains one subgroup of order 11, which is normal in $J$ (one can use the theory of Sylow to see the latter fact), Lemma 3.1.3(ii) yields

$$\text{Hom}(T, J) = \text{Hom}(T, Z_{11})$$

and hence

$$\text{Hom}^{**}(T, J) = \emptyset.$$

Using Lemmas 3.1.2(i) and 3.1.3(iii), we get

$$|\text{Hom}^1(T, M_{11})| = |\text{Hom}^{**}(T, M_{11})| - |\frac{11}{11}|M_{11}| |\text{Hom}^{1}(T, L_2(11))|$$

and then using the results computed in the previous subsection together with (3.8) and [5, Proposition 6.1 (iii)], we get Proposition 3.4.2(ii).
Finally, let us treat the case where \((p_1, p_2, p_3) = (5, 5, 5)\). By Lemma 3.1.2(i) and Proposition 3.1.4,

\[
\text{Hom}^{**}(T, M_{11}) = \text{Hom}^1(T, M_{11}) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, L_2(11)^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, (A_6 \cdot 2)^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, S_5^g).
\]

The previous equation can be rewritten as

\[
\text{Hom}^{**}(T, M_{11}) = \text{Hom}^1(T, M_{11}) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^1(T, L_2(11)^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, J^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^1(T, A_6^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, A_5^g) \\
\cup \bigcup_{g \in M_{11}} \text{Hom}^{**}(T, \tilde{A}_5^g),
\]

since there are two classes of \(A_5\) in \(M_{11}\), by Lemma 3.1.5.

In order to compute the size of \(\text{Hom}^1(T, M_{11})\), we need the following observations:

(i) If \(g\) is an element of \(M_{11}\) such that \(J^g \neq J\), then \(J \cap J^g\) is cyclic. Indeed, any proper subgroup of \(J\) is cyclic. Hence one can conclude that

\[
\text{Hom}^{**}(T, J) \cap \text{Hom}^{**}(T, J^g) = \emptyset.
\]

(ii) From the subgroup structure of \(A_5\) (see [2]), one knows that

\[
\text{Hom}^{**}(T, A_5) = \text{Hom}^1(T, A_5).
\]

Equipped with this information and Lemma 3.1.5, we can write:

\[
|\text{Hom}^{**}(T, M_{11})| = |\text{Hom}^1(T, M_{11})| \\
+ |M_{11} : L_2(11)||\text{Hom}^1(T, L_2(11))| \\
+ |M_{11} : J||\text{Hom}^{**}(T, J)| \\
+ |M_{11} : A_6 \cdot 2||\text{Hom}^1(T, A_6)| \\
+ |M_{11} : S_5||\text{Hom}^1(T, A_5)| \\
+ |M_{11} : A_5||\text{Hom}^1(T, A_5)|.
\]
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Now by [5, Proposition 6.1 (iii)] we have

$$|	ext{Hom}^1(T, L_2(11))| = 16|L_2(11)|$$

$$|	ext{Hom}^1(T, A_6)| = 16|A_6|$$

and [5, Proposition 5.1] gives us

$$|	ext{Hom}^{**}(T, J)| = 1320$$

$$|	ext{Hom}^1(T, A_5)| = 2|A_5|$$

Finally, 3.8 yields

$$|	ext{Hom}^{**}(T, M_{11})| = 594000$$

and hence Proposition 3.4.2 (i) follows.

3.4.3 Conclusion

We finally determine the probability that a randomly chosen homomorphism in $\text{Hom}(T, M_{11})$ is surjective.

Theorem 3.4.3

The following assertions hold

$$P_{5,5,5} = \frac{24}{603505}|M_{11}|$$

$$P_{11,11,11} = \frac{30}{270721}|M_{11}|$$

Proof. By 3.1.2(ii) we have

$$P = \frac{|	ext{Hom}^1(T, M_{11})|}{|	ext{Hom}(T, M_{11})|}$$

and so the results follow from Propositions 3.4.1 and 3.4.2. $\square$
Bibliography


