Topological groups: an introduction and first examples

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Abstract

In the first part, this document presents the concepts of topological groups and some elementary properties. In the second part, we give some examples of topological groups which are matrix groups. In the third part, we explain how to equip the set of homomorphisms between two abelian groups with a topological group structure. In the last part, we explain how to define a topological monoid structure on the set of non empty compact subsets of a given abelian topological group.

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1 Introduction

The goal of this document is to give an introduction about topological groups and some examples. Some results are inspired from [Bou60]. In section 2, we recall some concepts about topological spaces and uniform spaces. We also define some uniform structures on the set of maps between two uniform spaces. This is defined in [Bou74] TG10. Finally we prove a result about the hyperspace of a uniform space.

In section 3, we give the definition of a topological space and some elementary properties. For example we show that the topology is entirely defined by the filter of neighborhoods at the identity. In the second part of section 3, we define the quotient topology of a quotient group and prove that it is indeed a topological group.

In the third part of section 3, we define the concept of topological ring and topological field. Then, we prove that given a topological field and an integer \(n\), the ring of \(n \times n\) matrices over the field is a topological ring and the group of invertible \(n \times n\) matrices over the field is a topological group. We give some examples of such topological groups and subgroups. In the fourth part of section 3 we prove that the topology of a topological group comes from one or two uniform structures. These two uniform structures coincide if the group is abelian or compact. In the fifth part of section 3, we give different ways to define a uniform structure on the set of continuous homomorphisms between an arbitrary group and an abelian group. Finally we explain how to defined a topological group structure on the set of non empty compact subsets of a given topological group for which the two uniform structures coincide.

2 Topological concepts

In this section, we recall some fundamental concepts about topological spaces.

Notations

If \((X, \tau)\) is a topological space and \(x \in X\), we write \(\mathcal{V}(x)\) for the set of neighborhoods of \(x\).

A topological space is called compact if for every open covering, there exists a finite subcovering.

Proposition 2.1

Let \((X, \tau)\) and \((X', \tau')\) be two topological spaces. A map \(f : X \to X'\) is continuous if and only if for every \(x \in X\) and every \(V' \in \mathcal{V}(f(x))\) there exists \(V \in \mathcal{V}(x)\) such that, \(y \in V\) implies \(f(y) \in V'\).

Proposition 2.2

Let \((X, \tau)\) be a topological space. Suppose for every \(x \in X\), there exists the smallest neighborhood of \(x, V_x\). Then the family \(\{V_x\}_{x \in X}\) is a basis for the topology \(\tau\).

Remark 2.3

The condition of this proposition is satisfied if and only if the interior of the intersection of all neighborhoods of \(x\) contains \(x\). This is the case in particular if \(X\) is finite.

Now, we recall a lemma that explains how to define a topology on a set \(X\) if a particular family of filters at each point is given.

Lemma 2.4

Let \(X\) be a set and assume that for each \(x \in X\), we have a filter \(\mathcal{W}(x)\) on \(X\) satisfying:

1. \(\forall x \in X\) and \(\forall V \in \mathcal{W}(x), x \in V\);
2. \(\forall x \in X\) and \(\forall V \in \mathcal{W}(x), \exists U \in \mathcal{W}(y)\) s.t. \(\forall y \in U, V \in \mathcal{W}(y)\).

In this case, there exists a unique topology \(\tau\) on \(X\) such that for all \(x \in X\), the filter \(\mathcal{W}(x)\) is the set of neighborhoods of \(x\). This topology is given by

\[
\tau = \{ O \subset X | \forall x \in O, O \in \mathcal{W}(x) \}
\]

Proof. See [Bou71] I.3

We recall two propositions about the product. The product of topological spaces is always equipped with the product topology.

Proposition 2.5

Let \(X\) be a topological space, \(n \in \mathbb{N}\) and \(\sigma\) a permutation of \(n\) elements. Then the map

\[
\begin{array}{ccc}
X^n & \longrightarrow & X^n \\
(x_1, \ldots, x_n) & \longmapsto & (x_{\sigma(1)}, \ldots, x_{\sigma(n)})
\end{array}
\]

is continuous.
Proposition 2.6
Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be topological spaces and let $f_1 : X_1 \to Y_1, \ldots, f_n : X_n \to Y_n$ be continuous maps. The map $f_1 \times \cdots \times f_n$ defined by

$$X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n$$

$$(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n))$$

is continuous. This map is called the product map of the $f_i$.

Definition
Let $(X, \tau)$ be a topological space, and let $\sim$ be an equivalence relation on $X$. We can equip the quotient space $X/\sim$ with a topology $\tau_{\sim}$ defined by, $V \in \tau_{\sim} \iff \pi^{-1}(V) \in \tau$, where $\pi : X \to X/\sim$ is the quotient map sending $x \in X$ to its equivalence class.

So every open $V \in \tau_{\sim}$ can be be written as $\pi(U)$ for $U \in \tau$ but the converse is not true in general.

Proposition 2.7
Let $(X, \tau)$ and $\sim$ be as above and let $(X', \tau')$ be another topological space. For every map $f : X \to X'$ such that $x \sim y$ implies $f(x) = f(y)$, there exists a unique map $\tilde{f} : X/\sim \to X'$ such that $\tilde{f} \circ \pi = f$. Moreover, $f$ is continuous if and only if $\tilde{f}$ is continuous.

Proof.
We can and we must define $\tilde{f}(\pi(x)) = f(x)$. For $U' \in \tau'$, we have the following equivalences.

$$f^{-1}(U') \in \tau \iff (\tilde{f} \circ \pi)^{-1}(U') \in \tau \iff \pi^{-1}(\tilde{f}^{-1}(U')) \in \tau \iff \tilde{f}^{-1}(U') \in \tau_{\sim}.$$  

So $f$ is continuous if and only if $\tilde{f}$ is continuous. \hfill \Box

Proposition 2.8
Let $X$ be a topological space. If $F \subset X$ is closed and $C \subset X$ is compact, then $F \cap C$ is compact.

Proof. Let $\{O_i\}_{i \in I}$ be a family of open sets such that $F \cap C \subset \bigcup_{i \in I} O_i$. By hypothesis, $X \setminus F$ is open. We have

$$C \subset (F \cap C) \cup (X \setminus F) \subset \bigcup_{i \in I} O_i \cup (X \setminus F).$$

Since $C$ is compact, there exist $O_1, \cdots, O_n \in \{O_i\}_{i \in I}$ such that $C \subset \bigcup_{j=1}^n O_j \cup (X \setminus F)$. So we have

$$F \cap C \subset \bigcup_{j=1}^n (O_j \cup (X \setminus F)) \cap F = \bigcup_{j=1}^n O_j \cap F \subset \bigcup_{j=1}^n O_j.$$  

\hfill \Box

Definition
A topological space is called locally compact if for each point there exists a basis of compact neighborhoods.

Proposition 2.9
Let $X$ be a topological space and let $X' \subset X$. The following are equivalent

i) There exists a closed subset $F$ and an open subset $O$ such that $X' = O \cap F$.

ii) For every $x \in X'$ there exists a neighborhood $V_x$ of $x$ such that $V_x \cap X'$ is closed in $V_x$.

A subset $X'$ satisfying these conditions is called locally closed in $X$.

Proof. See [Bou71] TG I.20 §3 \hfill \Box

Remarks 2.10
A closed subset of a topological space is locally closed and an open subset of a topological space is locally closed.

Proposition 2.11
Let $X$ be a locally compact topological space and let $X' \subset X$ be a locally closed topological. Then $X'$ is locally compact for the induced topology.
Remark 2.14

2.1 Some concepts about uniform spaces

Proof. Let $x \in X'$ and $W$ be a neighborhood of $x$ for the induced topology and let us give a neighborhood $W' \subset W$ of $x$ which is compact for the induced topology. By definition of the induced topology, $W = U \cap X'$ for $U$ a neighborhood of $x$. Let $V$ be a neighborhood of $x$ such that $V \cap X'$ is closed in $V$. Let $U_1 := U \cap V$. The set $U_1$ is a neighborhood of $x$ and $X' \cap U_1$ is closed in $U_1$. Let $U_2 \subset U_1$ be a compact neighborhood of $x$. The set $X' \cap U_2 = X' \cap U_1 \cap U_2$ is closed in $U_1 \cap U_2 = U_2$ and so is compact by Proposition 2.8. So $W' := U_2 \cap X'$ is a neighborhood of $x$ for the induced topology which satisfies $W' \subset W$ and is compact in $X'$.

This proof is based on [Bou71] I.66.

Example 2.12

For every $0 < m \in \mathbb{N}$, the topological space $\mathbb{R}^m$ with the usual metric is locally compact. So each closed subset of $\mathbb{R}^m$ and each open subset of $\mathbb{R}^m$ is locally compact.

Proposition 2.13

The set $\{(p,q) \in \mathbb{Q} \times \mathbb{Q} | p^2 + q^2 = 1\}$ is dense in the set $\{(a,b) \in \mathbb{R} \times \mathbb{R} | a^2 + b^2 = 1\}$, where the topology is the induced from the usual of $\mathbb{R}$.

Proof. Let $(a,b) \in \mathbb{R}^2$ such that $a^2 + b^2 = 1$. There exists $t \in \mathbb{R}$ such that $(a,b) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. Let $\epsilon > 0$ and let us find an element $(p,q) \in \mathbb{Q} \times \mathbb{Q}$ such that $p^2 + q^2 = 1$ and $||(a,b) - (p,q)|| < \epsilon$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ and by the fact that the functions, $t \mapsto \frac{1-t^2}{1+t^2}$ and $t \mapsto \frac{2t}{1+t^2}$ are continuous, there exists $t' \in \mathbb{Q}$ such that $|\frac{1-t'^2}{1+t'^2} - \frac{1-t^2}{1+t^2}| < \frac{\epsilon}{2}$ and $|\frac{2t'}{1+t'^2} - \frac{2t}{1+t^2}| < \frac{\epsilon}{2}$. So the element $(\frac{1-t'^2}{1+t'^2}, \frac{2t'}{1+t'^2}) \in \{(p,q) \in \mathbb{Q} \times \mathbb{Q} | p^2 + q^2 = 1\}$ is the required element.

2.1 Some concepts about uniform spaces

We assume the reader is familiar with the definition of a uniform space. One can find this in [Mer10]. We will keep the same notations as in [Mer10]. We must add some other definitions.

Remark 2.14

If $X$ is a set and $\{\mathcal{U}_i\}_{i \in I}$ is a non-empty family of uniform structures on $X$, there exists a uniform structure on $X$ which is finer than each $\mathcal{U}_i$ in the family $\{\mathcal{U}_i\}_{i \in I}$. Indeed, define $\mathcal{U} = \cap_{i \in I} \mathcal{U}_i$ and check that it is a uniform structure on $X$. For any set, there exists at least the discrete uniform structure, so the family will not be empty in general.

Definition

Let $(X, \mathcal{U}_X)$ be a uniform space and let $X' \subset X$. We define $\mathcal{U}_{X'}$ to be the least fine uniform structure on $X'$ such that the injection $X' \rightarrow X$ is uniformly continuous. So, the entourages in $\mathcal{U}_{X'}$ are the sets $X' \times X' \cap V$ for $V \in \mathcal{U}_X$.

Remark 2.15

If $f : X \rightarrow Y$ is a uniformly continuous map. Then for every $X' \subset X$, the restriction of $f$ to $X'$ is also uniformly continuous. And for every $Y' \subset Y$ that contains the image of $f$, $f : X \rightarrow Y'$ is also uniformly continuous.

Definition

Let $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$ be two uniform spaces. We define $\mathcal{U}_{X \times Y}$ to be the least fine uniform structure on $X \times Y$ such that the projections on $X$ and on $Y$ are uniformly continuous maps. So we may take as a basis for the uniform structure $\mathcal{U}_{X \times Y}$ the family of sets $V \times W := \{((x_1, y_1), (x_2, y_2)) \in (X \times Y) \times (X \times Y) | (x_1, x_2) \in V, (y_1, y_2) \in W\}$

for $V \in \mathcal{U}_X$ and $W \in \mathcal{U}_Y$. This is an abuse of notation but we understand in the context. From now on, every product of uniform spaces will be equipped with this uniform structure.

Remark 2.16

If $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ are two uniformly continuous maps, then the map $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is uniformly continuous.

We need to introduce further notations. Let $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$ be two uniform spaces.

Notation

Let $V \in \mathcal{U}_X$, $W \in \mathcal{U}_Y$, $(x,y) \in X \times Y$, $A \subset X$ and $B \subset Y$. We write

$(V \times W)((x, y)) := \{(x', y') \in X \times Y | (x, x') \in V, (y, y') \in W\}$, \quad $(V \times W)[A \times B] := \bigcup_{a \in A, b \in B} (V \times W)((a, b))$. 

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Now we consider a set $X$ and a uniform space $(Y, \mathcal{U})$.

**Notation**

We write $\text{Set}(X,Y)$ for the set of functions $X \to Y$.

We will equip $\text{Set}(X,Y)$ with a uniform structure. For $V \in \mathcal{U}$ let $R(V) \subset \text{Set}(X,Y) \times \text{Set}(X,Y)$ be the set of ordered pairs $(f_1,f_2)$ of functions such that for all $x \in X$, $(f_1(x),f_2(x)) \in V$.

**Proposition 2.17**

$\{R(V)\}_{V \in \mathcal{U}}$ is a basis for a uniform structure on $\text{Set}(X,Y)$.

**Proof.** We prove that the set $\{R(V)\}_{V \in \mathcal{U}}$ satisfies the conditions of Remarks 3.3 in [Mer10]. i) For all $V \in \mathcal{U}$, all $f \in \text{Set}(X,Y)$, and all $x \in X$, $(f(x),f(x)) \in V$. On the other hand for every $V_1, V_2 \in \mathcal{U}$, we have the following properties. $R(V_1 \cap V_2) = R(V_1) \cap R(V_2)$, $R(V_1 \circ V_2) = R(V_1) \circ R(V_2)$ and $R(V_1^{-1}) = R(V_1)^{-1}$. This proves the conditions ii), iii), iv) in Remarks 3.3 in [Mer10].

The uniform space thus defined is written $\text{Set}{\mathcal{U}}(X,Y)$ and is called the **uniform convergence structure**. We are using the notations from [Bou74] TG X §I. Let $\mathfrak{S} \subset \mathcal{P}(X)$ be a family of subsets of $X$. We define the **Sigma-convergence uniform structure** on $\text{Set}(X,Y)$ to be the least fine uniform structure on $\text{Set}(X,Y)$ such that for every $A \in \mathfrak{S}$ the restriction to $\text{Set}(X,Y) \to \text{Set}(A,Y)$ is uniformly continuous. We write this uniform space by $\text{Set}{\mathfrak{S}}(X,Y)$. One remarks that a sub-basis for this uniform structure can be chosen to be the family of sets $R(A,V) := \{(f_1,f_2) \in \text{Set}(X,Y) \times \text{Set}(X,Y) \mid \forall x \in A, (f_1(x),f_2(x)) \in V\}$, for $A \in \mathfrak{S}$ and $V \in \mathcal{U}$. This means that a basis of $\text{Set}{\mathfrak{S}}(X,Y)$ is the set of finite intersections of entourages of the form $R(A,V)$ for $A \in \mathfrak{S}$ and $V \in \mathcal{U}$.

**Remarks 2.18**

To prove that a map $\text{Set}{\mathfrak{S}}(X,Y) \to \text{Set}{\mathfrak{S}}(X',Y')$ is uniformly continuous, it is sufficient to consider the sub-basis of entourages $R(A',V')$ because after one can take intersections. If $V_1, V_2 \in \mathcal{U}$ and $V_1 \subset V_2$ then $R(A,V_1) \subset R(A,V_2)$. So it is sufficient to consider the $V$ in a basis for the uniform structure $\mathcal{U}$.

**Proposition 2.19**

The canonical bijection

$\phi : \text{Set}{\mathfrak{S}}(X,Y) \times \text{Set}{\mathfrak{S}}(X,Y') \longrightarrow \text{Set}{\mathfrak{S}}(X,Y \times Y')$

$\phi(f_1,f_2) \mapsto (f_1(x),f_2(x))$

is uniformly continuous.

**Proof.** Let $A \in \mathfrak{S}$, $V \in \mathcal{U}$ and $V' \in \mathcal{U}'$. We claim that if $(f_1,f_1'),(f_2,f_2') \in R(A,V) \times R(A,V')$ then $\phi(f_1,f_1'),\phi(f_2,f_2') \in R(A,V \times V')$. Indeed, we have by hypothesis that $\forall x \in A, (f_1(x),f_2(x)) \in V$ and $(f_1'(x),f_2'(x)) \in V'$. So,

$\forall x \in A, \phi(f_1,f_2)(x) \in V$ and $\phi(f_1',f_2')(x) \in V'$.

**Proposition 2.20**

Let $h : (Y, \mathcal{U}) \to (Y', \mathcal{U}')$ be a uniformly continuous map. Then the left composition

$\text{Set}{\mathfrak{S}}(X,Y) \longrightarrow \text{Set}{\mathfrak{S}}(X,Y')$

$f \longmapsto h \circ f$

is uniformly continuous.

**Proof.** Let $A \in \mathfrak{S}$ and let $V' \in \mathcal{U}'$. We will consider the entourage $R(A,V')$. Since $h$ is uniformly continuous, there exists $V \in \mathcal{U}$ such that $(y_1,y_2) \in V$ implies $(h(y_1),h(y_2)) \in V'$. We claim that if $(f_1,f_2) \in R(A,V)$ then $(h \circ f_1,h \circ f_2) \in R(A,V')$. Indeed, let $x \in A$. $(f_1(x),f_2(x)) \in V$. So, $(h \circ f_1(x),h \circ f_2(x)) \in V'$.

Now we give an interesting result about the hyperspace of a uniform space. Let us recall (one can find this in [Mer10] 4.3) that given a uniform space $(X, \mathcal{U})$, one can define a uniform structure $\mathcal{U}^1$ on the family $\mathcal{K}X$ of non empty compact subsets of $X$. A basis for this uniform structure is given by the family of sets $Q(V) := \{(A,B) \in \mathcal{K}(X) \times \mathcal{K}(X) \mid A \subset V[B] \text{ et } B \subset V[A]\}$, for $V \in \mathcal{U}$.
Thus, If \(G\) and \(H\) are groups and \(G/H\) is a topological group, it follows by induction that for every \(n \in \mathbb{N}\)

The Cartesian product

\[ X \times X' \rightarrow X' (X \times X') \]

\((A, B) \rightarrow A \times B\)

is uniformly continuous.

**Proof.** By Tychonoff’s theorem (the product of compacts is compact), we know that the map is well defined. Let us prove that it is uniformly continuous. It is sufficient to consider an entourage in \(X \times X’\) of the form \(Q(V \times V')\) for \(V \in \mathcal{V}_X, V' \in \mathcal{V}_{X'}\). We claim that if \(((A_1, B_1), (A_2, B_2)) \in Q(V) \times Q(V')\) then \((A_1 \times B_1, A_2 \times B_2) \in Q(V \times V')\). Indeed, let \(((A_1, B_1), (A_2, B_2)) \in Q(V \times V')\). We have

\((A_1, A_2) \in Q(V)\) i.e. \((\forall a_1 \in A_1, \exists a_2 \in A_2\) such that \((a_2, a_1) \in V\) and \(\forall a_2 \in A_2, \exists a_1 \in A_1\) such that \((a_1, a_2) \in V\), and

\((B_1, B_2) \in Q(V')\) i.e. \((\forall b_1 \in B_1, \exists b_2 \in B_2\) such that \((b_2, b_1) \in V'\) and \(\forall b_2 \in B_2, \exists b_1 \in B_1\) such that \((b_1, b_2) \in V'\).

So

\[ \forall (a_1, b_1) \in A_1 \times B_1, \exists (a_2, b_2) \in A_2 \times B_2\] such that \((a_2, a_1) \in V, (b_2, b_1) \in V'\),

and

\[ \forall (a_2, b_2) \in A_2 \times B_2, \exists (a_1, b_1) \in A_1 \times B_1\] such that \((a_1, a_2) \in V, (b_1, b_2) \in V'\).

Thus

\[ A_1 \times B_1 \subset (V \times V') [A_2 \times B_2]\] and \(A_2 \times B_2 \subset (V \times V') [A_1 \times B_1]\).

Therefore, \((A_1 \times B_1, A_2 \times B_2) \in Q(V \times V')\). This proves that the Cartesian product is uniformly continuous. \(\square\)

3 Topological groups

3.1 Definition and some properties

**Notations**

If \(G\) is a group and \(\emptyset \neq H, K \subset G\), we write \(HK := \{hk \mid h \in H, k \in K\}\) and \(H^{-1} = \{h^{-1} \mid h \in H\}\). \(H^n = \{e\}\).

For \(n \in \mathbb{N}\setminus\{0\}\), \(H^n = \underbrace{H \cdot H \cdots H}_{n\text{ times}}\). For \(n \in \mathbb{Z}\) \(n < 0\), \(H^n = (H^{-n})^{-1}\). If \(H = H^{-1}\), we say that \(H\) is **symmetric**.

If \(G_1 \subset G\), we write \(G_1 < G\) to say \(G_1\) is a subgroup (not necessarily strict) of \(G\) and we write \(G_1 \triangleleft G\) to say \(G_1\) is a normal subgroup (not necessarily strict) of \(G\).

**Remarks 3.1**

If \(V, V_1, V_2 \subset G\) then \((V_1 \cap V_2)^{-1} = V_1^{-1} \cap V_2^{-1}\). \((V_1 V_2)^{-1} = V_2^{-1} V_1^{-1}\). If \(N \triangleleft G\), then \(VN = NV\).

**Definition**

Let \((G, \cdot)\) be a group endowed with a topology \(\tau\). We say that \(((G, \cdot), \tau)\) (or simply \(G\)) is a **topological group** if the product \(\cdot : G \times G \rightarrow G\) and the inverse \(G \rightarrow G\) are continuous maps.

**Remark 3.2**

It follows that for every \(g \in G\) the maps \(x \mapsto xg, x \mapsto gx, x \mapsto x^{-1}\) are homeomorphisms.

**Remark 3.3**

If \(G\) is a topological group, it follows by induction that for every \(n \in \mathbb{N}\), the multiplication,

\[
G^n \rightarrow G \\
(g_1, \cdots, g_n) \rightarrow g_1 \cdots g_n
\]

is continuous.

Indeed, if \(\prod : G \times G \rightarrow G\) denotes the multiplication, one can consider the maps

\[
\prod \times \text{Id}_{G}^{n-2} : G^n \rightarrow G^{n-1} \\
(g_1, g_2, \cdots, g_n) \rightarrow (g_1 g_2, g_3, \cdots, g_n)
\]

with is continuous by Proposition 2.6. The multiplication \(G^n \rightarrow G\) can be express as the composition of such continuous maps.

\[
G^n \rightarrow G^{n-1} \rightarrow \cdots \rightarrow G^2 \rightarrow G.
\]
Definitions
Let \((G, \tau)\) and \((G', \tau')\) be two topological groups. A map \(f : G \to G'\) is called a morphism of topological groups if \(f\) is a homomorphism and continuous. Of course, the identity is always a morphism of topological groups and the composition of two morphisms of topological groups is still a morphism of topological groups. So we can define \(\text{Top-grp}\) as the category of topological groups. The objects are the topological groups and the morphisms are the morphisms of topological groups.

Now, we will study some properties of the neighborhoods in a topological group. Let \(((G, \cdot), \tau)\) be a topological group with identity \(e\) and let \(\mathcal{B}\) be the filter of the neighborhoods of \(e\). We write \(\mathcal{V}(g)\) for the filter of neighborhoods of \(g\), where \(g \in G\). We write \(\text{inv} : G \to G\) for the map \(g \mapsto g^{-1}\).

With the remark 3.2 and Proposition 2.1, we deduce:

**Lemma 3.4**
The elements of \(\mathcal{V}(g)\) are precisely the subsets \(gU\) for \(U \in \mathcal{B}\) or equivalently, the subsets \( Ug \) for \( U \in \mathcal{B}\).

So the topology \(\tau\) is uniquely determined by \(\mathcal{B}\). Applying this fact we deduce:

**Proposition 3.5**
Let \(G_1\) and \(G_2\) be two topological groups with identity \(e, e'\) and let \(f : G_1 \to G_2\) be a homomorphism of groups. Then \(f\) is continuous if and only if for every \(V' \in \mathcal{V}(e')\) there exists \(V \in \mathcal{V}(e)\) such that \(x \in V\) implies \(f(x) \in V'\).

**Proof.** The condition is obviously necessary. Let us prove that it is sufficient using Proposition 2.1. Let \(g \in G_1\) and \(U' \in \mathcal{V}(f(g))\). By Lemma 3.4, there exists \(V' \in \mathcal{V}(f(g))\) such that \(U' = f(g)V'\). By hypothesis, there exists \(V \in \mathcal{V}(e)\) such that \(x \in V\) implies \(f(x) \in V'\). By Lemma 3.4, \(gV \in \mathcal{V}(g)\) and if \(y \in gV\) then \(f(y) \in f(g)V' = U'\), so \(f\) is continuous.

Now using the fact that \(\cdot : G \times G \to G\) is continuous at \((e, e)\) and the fact that \(\text{inv} : G \to G\) is continuous at \(e\), we have the following lemma.

**Lemma 3.6**
TG I: For all \(V \in \mathcal{B}\), there exists \(U \in \mathcal{B}\) such that \(UU \subset V\).

TG II: For all \(V \in \mathcal{B}\), there exists \(U \in \mathcal{B}\) such that \(U^{-1} \subset V\).

**Remark 3.7**
Let \(\mathcal{B}'\) be an arbitrary filter on \(G\) with \(\emptyset \notin \mathcal{B}'\). Then \(\mathcal{B}'\) satisfies TG I and TG II if and only if \(\mathcal{B}'\) satisfies TG I: For all \(V \in \mathcal{B}'\), there exists \(U \in \mathcal{B}\) such that \(UU^{-1} \subset V\).

Indeed,

- Assume TG I and TG II and let \(V \in \mathcal{B}'\). Let \(W \in \mathcal{B}'\) given by TG I such that \(WW \subset V\) and let \(U \in \mathcal{B}\) given by TG II such that \(U^{-1} \subset W \cap W^{-1} \subset \mathcal{B}'\). We have, \(UU^{-1} \subset (W \cap W^{-1})^{-1}(W \cap W^{-1}) = (W \cap W^{-1}) \subset W \subset V\).

- Assume TG I' and let \(V \in \mathcal{B}'\). One observes that for every \(U' \in \mathcal{B}'\), we have \(e \in U'\). Indeed, let \(W' \in \mathcal{B}'\) given by TG I' such that \(WW' \subset U'\). Since \(W' \neq \emptyset\), \(e \in W'W'^{-1} \subset U'\). Now, we deduce that \(V_1 \subset V_2 \subset \mathcal{B}'\) for all \(V_1, V_2 \in \mathcal{B}'\). Let \(U_1 \in \mathcal{B}'\) given by TG I' such that \(U_1U_1^{-1} \subset V\). Then \(U_1^{-1} \subset U_1U_1^{-1} \subset V\). So we have TG II. Let \(U \in \mathcal{B}'\) given by TG I' such that \(UU^{-1} \subset V\) and define \(U_2 = (U \cap U^{-1}) \in \mathcal{B}'\). Then we have, \(U_2U_2 = (U \cap U^{-1})(U \cap U^{-1}) \subset UU^{-1} \subset V\). So we have TG I.

**Remark 3.8**
For every \(V \in \mathcal{B}\) and every \(n \in \mathbb{N}\) there exists \(W \in \mathcal{B}\) such that \(W^n \subset V\). Indeed assume \(n > 0\). There exists \(m \in \mathbb{N}\) such that \(2^m \geq n\). Iterating TG I, there exists \(U \in \mathcal{B}\) such that \(U^{-n} \subset V\). Since \(e \in U\), we have \(U^n \subset U^{-n} \subset V\). Now, if \(n < 0\) take \(U \in \mathcal{B}\) such that \(U^{-n} \subset V\). Define \(W \in \mathcal{B}\) given by TG II such that \(W^{-1} \subset U\). Then we have \(W^n \subset V\). Moreover, we can have \(W\) symmetric. Indeed \(W \cap W^{-1} \in \mathcal{B}\) is symmetric and subset of \(W\).

Now using the fact that for every \(g \in G\) the map \(x \mapsto gxg^{-1}\) is continuous and sends \(e\) to \(e\), we deduce the following lemma.

**Lemma 3.9**
TG III: For all \(V \in \mathcal{B}\) and all \(g \in G\), \(gVg^{-1} \in \mathcal{B}\).

**Theorem 3.10**
Let \((G, \tau)\) be a finite topological group with identity \(e\) and let \(\mathcal{B}\) be the filter of neighborhoods of \(e\). There exists \(N \triangleleft G\) such that \(\mathcal{B} = \{V \subset G | N \subset V\}\).
Proof. Let us define $N := \bigcap_{V \in \mathcal{B}} V$. One remarks that $N \in \mathcal{B}$ because $G$ is finite. Let us prove that $N \triangleleft G$. One already knows that $e \in N$. Let $x, y \in N$. By TG I', there exists $U \in \mathcal{B}$ such that $UU^{-1} \subset N$. We have that $x \in U$ and $y \in U$, because $N \subset U$, so $xy^{-1} \in UU^{-1} \subset N$. Hence, $N < G$. Now let $g \in G$. By TG III, $gNg^{-1} \in \mathcal{B}$ and $g^{-1}Ng \in \mathcal{B}$. So, $N \subset gNg^{-1}$ and $N \subset g^{-1}Ng$. Hence $N = gNg^{-1}$.

**Lemma 3.11**
Let $(G, \tau)$ be a topological group. The closure $\overline{H}$ of a subgroup $H \subset G$ is a subgroup.

**Proof.** $\overline{H} \neq \emptyset$ of course. Let $x, y \in \overline{H}$ and let us prove that $xy^{-1} \in \overline{H}$. Let $W$ be a neighborhood of $xy^{-1}$. By Lemma 3.4, $W = xy^{-1}V$ for some $V \in \mathcal{B}$. Let $U \in \mathcal{B}$ given by TG I such that $UU^{-1} \subset V$. Let $U' \in \mathcal{B}$ given by TG III such that $U'y^{-1} \subset y^{-1}U$. By hypothesis, there exists $n_1 \in H \cap U'$ and $n_2 \in Uy \cap H$. So, $n_2^{-1} \in y^{-1}U^{-1} \cap H$. We have, $n_1n_2^{-1} \in H$ and

$$n_1n_2^{-1} \in xU'y^{-1}U^{-1} \subset xy^{-1}UU^{-1} \subset xy^{-1}V = W.$$ 

So $W \cap H \neq \emptyset$. □

**Lemma 3.12**
Let $(G, \tau)$ be a topological group. The closure $\overline{N}$ of a normal subgroup $N \triangleleft G$ is a normal subgroup.

**Proof.** We already know by Lemma 3.11 that $\overline{N} \subset G$. Let $g \in G$, $x \in \overline{N}$ and let us prove that $gxg^{-1} \in \overline{N}$. Let $gxg^{-1}V$ be a neighborhood of $gxg^{-1}$, with $V \subset \mathcal{B}$. Let $V' \subset \mathcal{B}$ be such that $V'y^{-1} \subset g^{-1}V$. By hypothesis, there exists $n \in xV' \cap N$. So $gng^{-1} \in gxV'g^{-1} \cap N$, since $N \triangleleft G$ and

$$gng^{-1} \in gxV'g^{-1} \subset gxg^{-1}V.$$ 

So $gxg^{-1}V \cap N \neq \emptyset$. □

**Lemma 3.13**
Let $(G, \tau)$ be a topological group. Let $H < K < G$ be two subgroups such that $[K, K] < H$. In this case $[\overline{K}, \overline{K}] < \overline{H}$.

**Proof.** Let $x \in [\overline{K}, \overline{K}]$ and let $V$ be a neighborhood of $e$ and let us prove that $xV \cap H \neq \emptyset$. We know by Lemma 3.11 that $\overline{H}$ is a subgroup, so it is sufficient to consider the case where $x = yzy^{-1}z^{-1}$ with $y, z \in \overline{K}$. Let us consider some neighborhoods of $e$ satisfying the following:

- $V_2, V_3$ such that $V_2V_3 \subset V$;
- $V_1, V'$ such that $V_1V'g^{-1}z^{-1} \subset g^{-1}z^{-1}V_2$;
- $V'' \subset V_3$ symmetric, $V''' \subset V'$ symmetric such that $V'''V''z \subset zV_1$.

By hypothesis, there exist $k_1 \in gv''' \cap K$ and $k_2 \in V''z \cap K$. So, $k_1^{-1} \in V'''y^{-1} \cap K$ and $k_2^{-1} \in z^{-1}V'' \cap K$. $k_1k_2k_1^{-1}k_2^{-1} \in [K, K] \subset H$ and we also have

$$k_1k_2k_1^{-1}k_2^{-1} \in yV'''V''zV'''y^{-1}z^{-1}V \subset yzV_1V'g^{-1}z^{-1}V_3 \subset yzy^{-1}z^{-1}V_2V_3 \subset yzy^{-1}z^{-1}V.$$ 

So $xV \cap H \neq \emptyset$. □

**Corollary 3.14**
Let $(G, \tau)$ be a topological group such that $\{e\}$ is closed and let $H < G$ be a subgroup. If $H$ is abelian, then $\overline{H}$ is also abelian.

**Proof.** We have by hypothesis $\{e\} = [H, H]$. By Lemma 3.13 and hypothesis,

$$[\overline{H}, \overline{H}] \subset \{e\} = \{e\}.$$ 

So $[\overline{H}, \overline{H}] = \{e\}$ and so $\overline{H}$ is abelian. □

**Corollary 3.15**
Let $(G, \tau)$ be a topological group such that $\{e\}$ is closed and let $H < G$ be a solvable subgroup. Then $\overline{H}$ is also solvable.
Proof. By hypothesis there exists \( n \in \mathbb{N} \) such that
\[
H = H_0 > H_1 > H_2 > \cdots > H_n = \{ e \},
\]
where \( H_i = [H_{i-1}, H_{i-1}] \). So, in particular we have
\[
H_i > H_{i+1} > [H_i, H_i], \forall i.
\]

By Lemma 3.13, we have
\[
\overline{H_{i+1}} > [\overline{H_i}, H_i].
\]

Let us define \( H'_0 := \overline{H_i}, H'_1 := [H', H'], H'_i := [H'_{i-1}, H'_{i-1}] \). We have
\[
H'_{m+1} > \overline{H_{m+1}} > [\overline{H_m}, H_m] \quad \text{(1)}
\]
Let us prove that the families \( H'_m, H'_m \) have the property that
\[
\{ e \} = \{ e \}.
\]

So, \( H'_n < \overline{H_n} \) for all \( i \). In particular we have
\[
H'_n < \overline{H_n} = \{ e \} = \{ e \}.
\]

Now we will study how, given a group \( G \), one can equip \( G \) with a topology \( \tau \) such that \( (G, \tau) \) is a topological group.

**Proposition 3.16**

Let \( G \) be a group with identity \( e \) and \( \emptyset \notin \mathcal{E} \subseteq \mathcal{P}(G) \) a filter satisfying TG I’ (or equivalently TG I and TG II).

Then we can equip \( G \) with a topology \( \tau_1 \) such that for all \( g \in G \) the set \( \{ gU \mid U \in \mathcal{E} \} \) is the set of neighborhoods of \( g \). With this topology, the left multiplication \( g \mapsto g'g \) is a homeomorphism for all \( g' \in G \).

**Proof.** Let us prove that the families \( \{ gU \mid U \in \mathcal{E} \} \) satisfy the conditions of Lemma 2.4.

\begin{itemize}
  \item \( \{ gU \mid U \in \mathcal{E} \} \) is a filter for every \( g \in G \).
  \item As in the remark 3.7, we see that \( e \in U \) for every \( U \in \mathcal{E} \). So \( g \in gU \) for every \( gU \in \{ gU \mid U \in \mathcal{E} \} \).
  \item Let \( gU \in \{ gU \mid U \in \mathcal{E} \} \) and let \( V \in \mathcal{E} \) such that \( VV \subseteq U \). We have \( gV \in \{ gU \mid U \in \mathcal{E} \} \) and for all \( y \in gV \), we have \( yV \subseteq gVV \subseteq gU \). So, for all \( y \in gV \), \( gU \in \{ gU \mid U \in \mathcal{E} \} \).
\end{itemize}

So the conditions of Lemma 2.4 are satisfied. Now let us prove that for every \( g' \in G \) the map \( g \mapsto g'g \) is continuous. Let \( g \in G \) and \( g'U \) be a neighborhood of \( g'g \) with \( U \in \mathcal{E} \) a neighborhood of \( e \). We have indeed that the neighborhood \( gU \) of \( g \) has the property that \( y \in gU \) implies that \( g'y \in g'gU \). So the condition of Proposition 2.1 is satisfied and so the left multiplication is continuous. Taking \( g^{-1} \) instead of \( g' \), we have that the inverse of the map \( g \mapsto g'g \) is continuous. So \( g \mapsto g'g \) is indeed a homeomorphism.

By an analogue argument we have:

**Proposition 3.17**

Let \( G \) be a group with identity \( e \) and \( \emptyset \notin \mathcal{E} \subseteq \mathcal{P}(G) \) a filter satisfying TG I’ (or equivalently TG I and TG II).

Then we can equip \( G \) with a topology \( \tau_r \) such that for all \( g \in G \) the set \( \{ Ug \mid U \in \mathcal{E} \} \) is set of neighborhoods of \( g \). With this topology, the right multiplication \( g \mapsto gg' \) is a homeomorphism for all \( g' \in G \).

**Proposition 3.18**

Let \( G \) be a group with identity \( e \) and \( \emptyset \notin \mathcal{E} \subseteq \mathcal{P}(G) \) a filter satisfying TG I’ (or equivalently TG I and TG II). Let \( \tau_1 \) and \( \tau_r \) as in Propositions 3.16 and 3.17. The map \( g \mapsto g^{-1} \) is a homeomorphism between \((G, \tau_1)\) and \((G, \tau_r)\).

**Proof.** Let \( g \in (G, \tau_1) \) and \( Ug^{-1} \) be a neighborhood of \( g^{-1} \) in \((G, \tau_1)\) with \( U \in \mathcal{E} \) a neighborhood of \( e \). One has indeed that \( gU \) is a neighborhood of \( g \in (G, \tau_1) \) and if \( y \in gU \) then \( y^{-1} \in Ug^{-1} \). So \((G, \tau_1) \to (G, \tau_r)\), \( g \mapsto g^{-1} \) is continuous by Proposition 2.1. Similarly the map \((G, \tau_r) \to (G, \tau_1)\), \( g \mapsto g^{-1} \) is also continuous so is a homeomorphism.
Lemma 3.19
Let \( G \) be a group with identity \( e \) and \( \emptyset \notin \mathcal{E} \subset \mathcal{P}(G) \) a filter satisfying TG I' (or equivalently TG I and TG II) and TG III. The topologies \( \tau_1 \) and \( \tau_r \) given by Propositions 3.16 and 3.17 are the same, and with this topology, \( G \) is a topological group.

Proof. To see that \( \tau_1 = \tau_r \), it is sufficient to see that for every \( g \in G \), \( \{ gU \mid U \in \mathcal{E} \} = \{ U'g \mid U' \in \mathcal{E} \} \). Let \( g \in G \) and \( U \in \mathcal{E} \). By TGIII \( gUg^{-1} = U' \) for some \( U' \in \mathcal{E} \). So we have, \( gU = gUg^{-1}g = U'g \in \{ U'g \mid U' \in \mathcal{E} \} \). Hence \( \{ gU \mid U \in \mathcal{E} \} \subset \{ U'g \mid U' \in \mathcal{E} \} \). Similarly \( \{ U'g \mid U' \in \mathcal{E} \} \subset \{ gU \mid U \in \mathcal{E} \} \). So, \( \tau_1 = \tau_r \).

Remarks 3.20
The conditions of Lemma 3.19 are in particular satisfied if \( N \triangleleft G \) and \( \mathcal{E} = \{ V \subset G \mid N \subset V \} \). We will write this topology \( \tau_N \). In this case the set \( G/N \) is a basis for the topology \( \tau_N \) (see Proposition 2.2) and since it is a partition of \( G \), \( (G, \tau_N) \) will never be connected excepted the case \( N = G \).

If \( N = \{e\} \), we have the discrete topology and if \( N = G \), we have trivial topology.

With Theorem 3.10, one deduces that giving a topology compatible with the group structure on a finite group is equivalent to giving a normal subgroup.

Proposition 3.21
Let \( G_1, G_2 \) be two groups and let \( N_1 \triangleleft G_1, N_2 \triangleleft G_2 \). If \( f : G_1 \to G_2 \) is a homomorphism such that \( f(N_1) \subset N_2 \) then \( f : (G_1, \tau_{N_1}) \to (G_2, \tau_{N_2}) \) is continuous.

Proof. With Proposition 3.5 and by definition of the topologies \( \tau_{N_1} \) and \( \tau_{N_2} \), it is sufficient to prove that for every set \( U' \subset G_2 \) such that \( N_2 \subset U \) there exists \( U \subset G_1 \) with \( N_1 \subset U \) such that \( f(U) \subset U' \). Take \( U = N_1 \) and the hypothesis on \( f \) gives the required inclusion.

Remark 3.22
The Condition on \( f \) in Proposition 3.21 is satisfied if we take \( N_1 = [G_1, G_1] \) and \( N_2 = [G_2, G_2] \). So, this allows us to define a full and faithful functor \( \text{Grp} \to \text{Top-grp} \) sending \( G \to (G, \tau_{[G,G]}) \) and a homomorphism to the same homomorphism.

Now, we give some properties about subsets of a topological groups.

Proposition 3.23
Let \((G, \tau)\) be a topological group, \( \emptyset \neq O \subset \tau \) be an open subset and \( \emptyset \neq A \subset G \). Then \( O\!A \) and \( AO \) are open subsets.

Proof. We can write
\[
O\!A = \bigcup_{a \in A} Oa \quad \text{and} \quad AO = \bigcup_{a \in A} aO.
\]
Every set \( Oa \) and \( aO \) is open as the image of an open subset by a homeomorphism and so \( AO \) and \( O\!A \) are open as a union of open subsets.

Remark 3.24
In particular, if one singleton is open then every singleton is open.

Proposition 3.25
Let \((G, \tau)\) be a topological group, \( \emptyset \neq F \subset G \) be a closed subset and \( \emptyset \neq B \subset G \) be a finite subset. Then \( FB \) and \( BF \) are closed subsets.

Proof. We can write
\[
FB = \bigcup_{b \in B} Fb \quad \text{and} \quad BF = \bigcup_{b \in F} bF.
\]
Every set \( Fb \) and \( bF \) is closed as the image of a closed subset by a homeomorphism and so \( FB \) and \( BF \) are closed subsets as a finite union of closed subsets.
Remark 3.26
In particular if one singleton is closed then every singleton is closed.

Lemma 3.27
Let \((G, \tau)\) be a topological group. An open subgroup \(H\) is closed.

Proof. We can write \(G \setminus H = \bigcup_{g \in H} gH\).

So \(G \setminus H\) is open as a union of open subsets.

Lemma 3.28
Let \((G, \tau)\) be a finite topological group. A closed subgroup \(K\) is open.

Proof. We can write \(G \setminus K = \bigcup_{g \in K} gK\).

So \(G \setminus K\) is closed as a finite union of closed subsets.

3.2 Quotient space of a topological group

We know that if we take a subgroup \(H\) of a given group \(G\) one can define an equivalence relation on \(G\) using \(H\). The quotient set (set equivalence class) is written \(G/H\). If \(G\) is a topological group one can define the quotient topology (see Definition 2) on \(G/H\). We will see in this case that the quotient map \(\pi : G \to G/H\) is open. This implies in particular that the set of open sets in \(G/H\) is the set of all \(\pi(O)\) for \(O\) an open set in \(G\). Moreover if \(H < G\) one knows that \(G/H\) has a group structure. In fact it will be a topological group.

Proposition 3.29
Let \(G\) be a topological group and \(H < G\). The quotient map \(\pi : G \to G/H\) is open, i.e. for every open set \(O\) in \(G\) the set \(\pi(O)\) is open in \(G/H\).

Proof. Let \(O\) be open in \(G\). We have \(\pi(O) = \{oH \mid o \in O\}\). To check that \(\pi(O)\) is open in \(G/H\) one must check that \(\pi^{-1}(\pi(O))\) is open in \(G\). We have

\[
\pi^{-1}(\pi(O)) = \{g \in G \mid gH \in \pi(O)\} = \{g \in G \mid \exists o \in O \text{ such that } gH = oH\} = \{g \in G \mid \exists o \in O, \exists h \in H \text{ such that } g = oh\} = OH,
\]

which is open by Proposition 3.23.

Remark 3.30
It follows that every family of open sets in \(G/H\) is the family of \(OH\) for \(O\) an open set in \(G\). In particular for \(xH \in G/H\) the set of neighborhoods of \(xH\) is the family of the \(VH\) where \(V\) is a neighborhood of \(x\).

Lemma 3.31
Let \(G\) be a topological group and let \(H < G\) be a normal subgroup. The space \(G/H\) is a topological group.

Proof. We use Proposition 2.1 to show that the multiplication and the inversion are continuous.

- Multiplication: Let \(aH, bH \in G/H\) and let \(V\) be a neighborhood of \(aHbH = abH\). By Remark 3.30, we can write \(V = OH\) for \(O\) a neighborhood of \(xy\). Since the multiplication is continuous in \(G\) there exists a neighborhood \(O_1\) of \(a\) and a neighborhood \(O_2\) of \(b\) such that \(O_1O_2 \subset O\). Again by Remark 3.30, \(O_1H\) is a neighborhood of \(aH\) and \(O_2H\) is a neighborhood of \(bH\). We have \(O_1HO_2H = O_1O_2H \subset OH = V\). So the multiplication is continuous.

- Inversion: Let \(xH \in G/H\) and let \(OH\) be a neighborhood of \((xH)^{-1} = x^{-1}H\) with \(O\) a neighborhood of \(x^{-1}\). By continuity of inversion in \(G\) there exists a neighborhood \(O'\) of \(x\) such that \(O'^{-1} \subset O\). Again by Remark 3.30, \(O'H\) is a neighborhood of \(xH\) and \((O'H)^{-1} = O'^{-1}H \subset OH\). So inversion is continuous.
3.3 Matrix groups

Notations
Let $F$ be a field and $n \in \mathbb{N}$. We write $\text{Mat}_n(F)$ for the ring of $n \times n$ matrices, with entries in $F$. We write $I_n$ for the identity matrix. If $A \in \text{Mat}_n(F)$ we write $A^t$ for the transpose of $A$. We write $\text{GL}_n(F) \subset \text{Mat}_n(F)$ for the set of invertible matrices. We write $\text{SL}_n(F) \subset \text{GL}_n(F)$ for the set of matrices of determinant 1. We write $O_n(F) \subset GL_n(F)$ for the set of the matrices $A$ such that $AA^t = I_n$. We write $SO_n(F) \subset O_n(F)$ for the set the matrices in $O_n(F)$ of determinant 1. We write $\det$ for the determinant map $\text{Mat}_n(F) \to F$.

Definition
A topological ring $((R, +, \cdot), \tau)$ is a ring endowed with a topology such that $(R, +)$ is a topological group and such that the multiplication is also continuous.

Remarks 3.32
If $R$ is a topological ring and $n \in \mathbb{N}$ the addition
\[
R^n \times R^n \to R^n
\]
\[(r_1, \ldots, r_n), (r_1', \ldots, r_n') \mapsto (r_1 + r_1', \ldots, r_n + r_n')
\]
is continuous. Indeed, one can express this map as the composition
\[
(r_1, \ldots, r_n), (r_1', \ldots, r_n') \mapsto (r_1, r_1', \ldots, r_n, r_n') \mapsto (r_1 + r_1', \ldots, r_n + r_n'),
\]
which are continuous by Proposition 2.5 and Proposition 2.6.

We also have the same result if we consider the coordinatewise multiplication instead of the addition.

Proposition 3.33
Let $R$ be a topological ring and $n \in \mathbb{N}$. The scalar product
\[
R^n \times R^n \to R
\]
\[(r_1, \ldots, r_n), (r_1', \ldots, r_n') \mapsto r_1 r_1' + \cdots + r_n r_n'
\]
is continuous.

Proof. The scalar product can by express as the composition of the maps
\[
(r_1, \ldots, r_n), (r_1', \ldots, r_n') \mapsto (r_1 r_1', \ldots, r_n r_n') \mapsto r_1 r_1' + \cdots + r_n r_n'
\]
which are both continuous. \qed

Definition
A topological field $((F, +, \cdot), \tau)$ is a field such that $((F, +, \cdot), \tau)$ is a topological ring and such that the inverse $F \setminus \{0\} \to F \setminus \{0\}$ is continuous.

Examples 3.34
1. $\mathbb{Q}$ with the usual topology induced from $\mathbb{R}$ is a topological field.
2. $\mathbb{R}$ with the usual topology is a topological field.
3. $\mathbb{C}$ with the usual topology is a topological field. Moreover, the conjugation $a + ib \mapsto a - ib$ is continuous.

Proposition 3.35
Let $((F, +, \cdot), \tau)$ be a topological field such that the topology is defined by a normal additive subgroup $(N, +) < (F, +)$ as in remark 3.20. In this case, the topology is discrete or trivial.

Proof. By definition of this topology, for every element $x \in F$, the set $x + N$ is the smallest neighborhood of $x$. Let us write the fact that the multiplication is continuous at $(y, 0) \mapsto 0$ for every $y$ via the condition in Proposition 2.1. The subset $N$ is a neighborhood of 0 and since $N$ is the smallest neighborhood of 0 and $y + N$ is the smallest neighborhood of $y$, we must have $(y + N)(N) \subset N$, i.e.
\[
yN + NN \subset N.
\]
(2)
If $N = \{0\}$ then we have the discrete topology. Suppose there exists $0 \neq h \in N$. Then writing (2) with $y = h^{-1}$ and $0 \cdot 0 \in NN$, we deduce that $1 \in N$. Now taking an arbitrary $y \in F$, we have $y \cdot 1 + 0 \cdot 0 \in N$. So $N = F$ and so the topology is trivial. \qed

Remark 3.36
With Theorem 3.10, we deduce that on a finite field, the only topologies compatible with the field structure are the trivial and the discrete topologies.
Let us consider $\mathbb{H}$ the $\mathbb{R}$-algebra with basis $1, i, j, k$ with the following rules:

$$ij = k = -ji \quad i^2 = j^2 = k^2 = -1.$$ 

So an element of $\mathbb{H}$ is uniquely written $x + iy + jz + kt$ with $x, y, z, t \in \mathbb{R}$. The multiplication rules imply that $ik = -j = -ki$ and $jk = i = -kj$. Now extending by $\mathbb{R}$-linearity, the multiplication of two elements in $\mathbb{H}$ is given by

$$(x_1 + iy_1 + jz_1 + kt_1)(x_2 + iy_2 + jz_2 + kt_2) =$$

$$(x_1x_2 - y_1z_2 - z_1y_2 - t_1t_2) + i(x_1y_2 + y_1x_2 + z_1t_2 + t_1z_2) + j(x_1z_2 - y_1t_2 + z_1x_2 + t_1y_2) + k(x_1t_2 + y_1z_2 - z_1y_2 + t_1x_2).$$

These elements are called the quaternions of Hamilton.

**Proposition 3.37**

In $\mathbb{H}$ we can define $\bar{x} : \mathbb{H} \to \mathbb{H}$ by $x + iy + jz + kt \mapsto x - iy - jz - kt$. With this, we can define $\|x + iy + jz + kt\| := (x + iy + jz + kt)(x + iy + jz + kt) = x^2 + y^2 + z^2 + t^2$. One remarks that for $q \in \mathbb{H}$, we have $q = 0 \iff \|q\| = 0$.

If $0 \neq q$, one can give an inverse of $q$ for the multiplication. This inverse is given by $\frac{1}{\|q\|^2}q$.

One can also check that for $q_1, q_2 \in \mathbb{H}$ we have $\|q_1q_2\| = \|q_1\|\|q_2\|$.

**Example 3.38**

We equip $\mathbb{H}$ with the usual topology of $\mathbb{R}^4$. One shows easily that $\mathbb{H}$ is a topological ring, $\bar{x} : \mathbb{H} \to \mathbb{H}$ is a homeomorphism, and $\| \cdot \| \to \mathbb{R}$ is continuous. So we conclude that the inverse $\mathbb{H} \setminus \{0\} \to \mathbb{H} \setminus \{0\}$ is also continuous.

**Remark 3.39**

If we consider the set $\{q \in \mathbb{H} \mid \|q\| = 1\}$ which is stable under multiplication, one remarks that it is a topological group. With the correspondence between $\mathbb{H}$ and $\mathbb{R}^4$ and the form of $\| \cdot \|$, we choose to call our set $\mathbb{S}^3$. This set is closed in $\mathbb{H}$ because it is the inverse image of $\{1\}$ by the continuous map $\| \cdot \| : \mathbb{H} \to \mathbb{R}$.

Given a topological field $((\mathbb{F}, +, \cdot), \tau)$, we will study the matrix ring $\text{Mat}_n(\mathbb{F})$. We identify $\text{Mat}_n(\mathbb{F})$ with $\mathbb{F}^{n^2}$ and endow $\text{Mat}_n(\mathbb{F})$ with the product topology.

**Remarks 3.40**

By definition of the product topology, the projection on an entry, the projection on a row or on a column, and the projection on a sub-matrix are continuous.

We deduce that multiplying a matrix by a scalar is continuous.

Taking a suitable permutation in Proposition 2.5, one remarks that sending a matrix to its transpose is a continuous map.

**Lemma 3.41**

Let $((\mathbb{F}, +, \cdot), \tau)$ be a topological field. The ring $(\text{Mat}_n(\mathbb{F}), +, \cdot)$ viewed as $\mathbb{F}^{n^2}$ with the product topology is a topological ring.

**Proof.** The addition of two matrices is continuous by the remark 3.32.

The multiplication can be seen as a product of continuous maps (see Proposition 2.6) as follows. We apply a suitable permutation of the $2n^2$ components (see Proposition 2.5), followed by a product of the scalar product (see Proposition 3.33). 

**Lemma 3.42**

Let $((\mathbb{F}, +, \cdot), \tau)$ be a topological field. The map $\det : (\text{Mat}_n(\mathbb{F}), +, \cdot) \to \mathbb{F}$ is continuous.

**Proof.** By induction on $n$. The case $n = 1$ is clear.

A determinant of an $n \times n$ matrix can be expressed as a sum of products of one entry by the determinant of an $n - 1 \times n - 1$ matrix, for example choosing the first column. So it is continuous as a composition of continuous maps (remarks 3.40, hypothesis of induction).

**Remark 3.43**

It follows from Lemma 3.42 and Proposition 2.6 that the map sending a matrix to the matrix of co-factors is continuous.

**Lemma 3.44**

Let $((\mathbb{F}, +, \cdot), \tau)$ be a topological field. The group $(\text{GL}_n(\mathbb{F}), \cdot)$ is a topological group.

**Proof.** We already know that $\cdot$ is continuous (Lemma 3.41). To see that inversion is continuous, remember that we have a formula to compute the inverse of a matrix; the inverse of the determinant times the transpose of the matrix of co-factors. This is a composition of continuous maps (see lemma 3.42 and remarks 3.40).
Example 3.45
Lemma 3.44 implies that $GL_n(\mathbb{R})$ is a topological group. This topological space is metrizable but not connected because $\det(GL_n(\mathbb{R})) = \mathbb{R} \setminus \{0\}$ which is not connected. Since $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$, one deduces that $GL_n(\mathbb{R})$ is open in $Mat_n(\mathbb{R})$. We also know that $GL_n(\mathbb{R})$ is locally compact (see Example 2.12).

We also know that $O_n(\mathbb{R})$ is a topological group. This group is closed in $GL_n(\mathbb{R})$ because it is the inverse image of $\{I_n\}$ by the continuous map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto AA^t$ (see Remarks 3.40). By Example 2.12 we know that $O_n(\mathbb{R})$ is locally compact. We also know that $SO_n(\mathbb{R})$ is a topological group. This group is closed in $GL_n(\mathbb{R})$ because it is the intersection of $O(\mathbb{R})$ and $\det^{-1}(\{1\})$. By Example 2.12 we know that $SO_n(\mathbb{R})$ is locally compact. This group is also closed in $O_n(\mathbb{R})$.

Example 3.46
We want to study $O_2(\mathbb{R})$ and $O_2(\mathbb{Q})$ but before studying the topological structure of these groups, we want to give an easy description as sets. More generally, let us consider a field $F$ and a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $O(F)$. i.e $a, b, c, d \in F$ and

$$AA^t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

In particular we have

$$ac + bd = 0 \quad ad - bc = \pm 1 \quad a^2 + b^2 = c^2 + d^2 = 1$$

Seen as a system of equations in $c$ and $d$, we can write

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}.$$ 

So,

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \mp b \\ \pm a \end{pmatrix}.$$ 

So the matrix $A$ is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ or of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ both with $a^2 + b^2 = 1$. Conversely, one shows that this condition is sufficient to be in $O_2(F)$. We conclude that

$$O_2(F) = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right), \left( \begin{array}{cc} a & b \\ b & -a \end{array} \right) \mid a, b \in F, a^2 + b^2 = 1 \right\}.$$ 

If the characteristic of $F$ is not 2 there are two different kinds of matrix in $O_2(F)$.

Now we want to study $SO_2(F)$. With the last result we have,

$$SO_2(F) = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \mid a, b \in F, a^2 + b^2 = 1 \right\}. $$

We claim $SO_2(F)$ is an abelian group. Indeed for $a, b, a', b' \in F$ such that $a^2 + b^2 = 1$ and $a'^2 + b'^2 = 1$, we have

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} = \begin{pmatrix} aa' - bb' & ab' + ba' \\ -ba' - ab' & bb' + aa' \end{pmatrix} = \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

Now we want study $[O_2(F), O_2(F)]$. Introducing $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, one observes that we can write $O_2(F) = SO_2(F) \cup \sigma SO_2(F)$. So, $[O_2(F), O_2(F)]$ is generated by the elements $[A, \sigma B], [\sigma A, B], [\sigma A, \sigma B]$, for $A, B \in SO_2(F)$. We have $\sigma^{-1} = \sigma$ and $AB = BA$ and $\sigma A \sigma = A^{-1}$. So,

$$[A, \sigma B] = A\sigma BA^{-1}(\sigma B)^{-1} = A\sigma BA^{-1}B^{-1} = A = A^{-1} = A\sigma A = A^2,$$

$$[\sigma A, B] = [B, \sigma A]^{-1} = B^{-2},$$

$$[\sigma A, \sigma B] = \sigma A\sigma BA^{-1}\sigma B^{-1} = \sigma A^{-1}BA^{-1}B = A^{-1}BA^{-1}B = A^{-2}B^2.$$ 

So we deduce, again using the fact that $SO_2(F)$ is abelian, that $[O_2(F), O_2(F)]$ is the set of all squares of elements of $SO_2(F)$, i.e.

$$[O_2(F), O_2(F)] = \left\{ \left( \begin{array}{cc} 2ab \\ -2ab \end{array} \right) \mid a, b \in F, a^2 + b^2 = 1 \right\}.$$
In particular \([O_2(\mathbb{F}), O_2(\mathbb{F})] \subset SO_2(\mathbb{F})\).

Now let us study the group \(O_2(\mathbb{R})\). This group is not connected because the map \(\text{det}(O_2(\mathbb{R})) = \{-1, 1\}\) which is not connected. We have the following nice result. The group \(SO_2(\mathbb{R})\) is isomorphic, as topological group, to \(S^1\) seen as the unit circle in \(\mathbb{C}\) with the usual topology and the multiplication of complex numbers as group operation. Let us consider the bijection

\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + ib.
\]

Observing the multiplication in \(SO_2(\mathbb{R})\) described above and remembering the multiplication in \(\mathbb{C}\), one deduces that the bijection is an isomorphism of groups. To see that it is a homeomorphism we can remember that these two topologies come from a metric. On \(SO_2(\mathbb{R})\), the distance between \(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\) and \(\begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix}\) is \(\sqrt{2(a-a')^2 + 2(b-b')^2} = \sqrt{2(a-a')^2 + (b-b')^2}\). In \(\mathbb{C}\) the distance between \(a + ib\) and \(a' + ib'\) is \(\sqrt{(a-a')^2 + (b-b')^2}\). So, under the bijection, the distance will be multiplied by \(\sqrt{2}\) or divided by \(\sqrt{2}\), so the bijection is clearly continuous in both directions.

Now we can say that \(SO_2(\mathbb{R})\) is compact (in particular closed since Hausdorff) and path connected and its fundamental group is \(\mathbb{Z}\). Seen only as topological spaces, the spaces \(SO_2(\mathbb{R})\) and \(\sigma SO_2(\mathbb{R})\) are homeomorphic. One deduces that the sets \(SO_2(\mathbb{R})\) and \(\sigma SO_2(\mathbb{R})\) are the two path connected components of \(O_2(\mathbb{R})\). Now we claim that \([O_2(\mathbb{R}), O_2(\mathbb{R})] = SO_2(\mathbb{R})\). We already know that \([O_2(\mathbb{R}), O_2(\mathbb{R})] \subset SO_2(\mathbb{R})\), so to prove this we must prove that for every \(x, y \in \mathbb{R}\) with \(x^2 + y^2 = 1\) there exist \(a, b \in \mathbb{R}\) with \(a^2 + b^2 = 1\) such that \(x = a^2 - b^2\) and \(y = 2ab\).

If \(y = 0\) one chooses \(a = 0\) or \(b = 0\) depending on the sign of \(x\). If \(y \neq 0\) then \(b \neq 0\) and we can write \(a = \frac{y}{b^2}\).

Replacing \(a\) in the equation with \(x\) we have

\[
x = \frac{y^2}{4b^2} - b^2 \implies 4b^2x = y^2 - 4b^4 \implies 4b^4 + 4xb^2 - y^2 = 0 \implies b = \frac{2x \pm \sqrt{4x^2 + 4y^2}}{2} = -x \pm \sqrt{x^2 + y^2}
\]

\[
\implies b^2 = -x \pm \frac{1}{2} \quad \text{or} \quad \frac{1}{2} - x.
\]

And \(a = \frac{y}{\pm 2\sqrt{1-x}} = \pm \frac{y}{\sqrt{2}\sqrt{1-x}}\). To summarize we have,

\[
a = \pm \frac{y}{\sqrt{2}\sqrt{1-x}} \quad \quad b = \pm \sqrt{-x - \frac{1}{2}}. \quad \quad (3)
\]

One checks,

\[
a^2 + b^2 = \frac{y^2}{2(1-x)} + \frac{1-x}{2} = \frac{y^2 + (1-x)^2}{2(1-x)} = \frac{y^2 + x^2 - 2x + 1}{2(1-x)} = \frac{1-2x+1}{2(1-x)} = \frac{2}{2(1-x)} = 1.
\]

In fact, using the bijection with \(S^1\) and the square root in \(\mathbb{C}\), one sees directly that a such matrix exists. We will use the formulas (3) in the following.

Now we want study the group \(O_2(\mathbb{Q})\). The subgroup \(SO_2(\mathbb{Q})\) is closed in \(O_2(\mathbb{Q})\) since it is the preimage by det of \(\{1\}\) which is closed. We claim \([O_2(\mathbb{Q}), O_2(\mathbb{Q})] \subset SO_2(\mathbb{Q})\). Indeed, let us consider the matrix

\[
X = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \in SO_2(\mathbb{Q}).
\]

To have \(X \in [O_2(\mathbb{Q}), O_2(\mathbb{Q})]\) there must exist \(a, b \in \mathbb{Q}\) such that \(a^2 - b^2 = \frac{3}{5}\) and \(2ab = \frac{4}{5}\) but in this case \(a\) and \(b\) must satisfy (3). This gives

\[
a = \pm \frac{4}{\sqrt{2}\sqrt{-\frac{2}{5}}} \quad \quad b = \pm \sqrt{-\frac{1}{2}}.
\]

which are not in \(\mathbb{Q}\). So, \(X \notin [O_2(\mathbb{Q}), O_2(\mathbb{Q})]\). But we have \([O_2(\mathbb{Q}), O_2(\mathbb{Q})] = SO_2(\mathbb{Q})\). Since \(SO_2(\mathbb{Q})\) is closed and \([O_2(\mathbb{Q}), O_2(\mathbb{Q})] \subset SO_2(\mathbb{Q})\), we have \([O_2(\mathbb{Q}), O_2(\mathbb{Q})] \subset SO_2(\mathbb{Q})\). To see the converse, let \(\begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in SO_2(\mathbb{Q})\) and let \(\epsilon > 0\) and let us prove that there exist \(p', q' \in \mathbb{Q}\) with \(p'^2 + q'^2 = 1\) and \(\begin{pmatrix} p' & q' \\ -q' & p' \end{pmatrix} \in [O_2(\mathbb{Q}), O_2(\mathbb{Q})]\) satisfying

\[
\begin{pmatrix} p & q \\ -q & p \end{pmatrix} - \begin{pmatrix} p' & q' \\ -q' & p' \end{pmatrix} = \begin{pmatrix} p^2 - p'^2 & q^2 - q'^2 \\ -q^2 - q'^2 & p^2 - p'^2 \end{pmatrix} \leq \epsilon.
\]

Using the formulas (3) we can find \(a, b \in \mathbb{R}\) such that \(a^2 + b^2 = 1\) and \(p = a^2 - b^2\) and \(q = 2ab\). Choose \(\epsilon' > 0\) such that \(|x_1 - y_1|, |x_2 - y_2| < \epsilon' \implies |x_1^2 - y_1^2| < \frac{\epsilon'}{7}\), \(2|x_1x_2 - y_1y_2| < \frac{\epsilon'}{7}\) By Proposition 2.13 there exist \(p', q' \in \mathbb{Q}\) such that \(p'^2 + q'^2 = 1\) and \(|p' - a| < \epsilon'\) and \(|q' - b| < \epsilon'\). Now we have
\[
\begin{pmatrix}
  p & q \\
  -q & q
\end{pmatrix} - \begin{pmatrix}
  p'^2 - q'^2 & 2pq' \\
  -2p'q' & p'^2 - q'^2
\end{pmatrix} = \begin{pmatrix}
  a^2 - b^2 & 2ab \\
  -2ab & a^2 - b^2
\end{pmatrix} - \begin{pmatrix}
  p'^2 - q'^2 & 2pq' \\
  -2p'q' & p'^2 - q'^2
\end{pmatrix} < \frac{\epsilon}{8} + \frac{\epsilon}{4} = \epsilon,
\]
as claimed.

**Example 3.47**

Let us consider the following set

\[
S = \left\{ \begin{pmatrix}
  x & y & -z & t \\
  -y & x & t & z \\
  z & -t & x & y \\
  -t & -z & -y & x
\end{pmatrix} \mid x, y, z, t \in \mathbb{R}, x^2 + y^2 + z^2 + t^2 = 1 \right\}
\]

One sees easily that \( S \subset O_4(\mathbb{R}) \) but we even have \( S \subset SO_4(\mathbb{R}) \). To see this we compute according to the first column.

\[
\det \begin{pmatrix}
  x & y & -z & t \\
  -y & x & t & z \\
  z & -t & x & y \\
  -t & -z & -y & x
\end{pmatrix} =
\]

\[
x (x(x^2 + y^2) + t(tx + yz) - z(ty - xz)) + y(y(x^2 + y^2) + t(-xz + ty) - z(-zy - tx))
\]

\[
+ z(y(tx + yz) - x(-zx + ty) - z(-z^2 - t^2)) + t(y(ty - xz) - x(-zy - tx) - t(-z^2 - t^2))
\]

\[= x^2 + y^2 + z^2 + t^2 = 1.\]

We claim that \( S \) is a subgroup of \( SO_4(\mathbb{R}) \) isomorphic (as topological group) to \( S^3 \) (see Remark 3.39) via the followings bijection

\[x + iy + jz + kt \longleftrightarrow \begin{pmatrix}
x & y & -z & t \\
-y & x & t & z \\
z & -t & x & y \\
-t & -z & -y & x
\end{pmatrix}\]

To see this let us look at the multiplication of two elements of \( S \).

\[
\begin{pmatrix}
x_1 & y_1 & -z_1 & t_1 \\
-y_1 & x_1 & t_1 & z_1 \\
z_1 & -t_1 & x_1 & y_1 \\
-t_1 & -z_1 & -y_1 & x_1
\end{pmatrix} \begin{pmatrix}
x_2 & y_2 & -z_2 & t_2 \\
-y_2 & x_2 & t_2 & z_2 \\
z_2 & -t_2 & x_2 & y_2 \\
-t_2 & -z_2 & -y_2 & x_2
\end{pmatrix} =
\]

\[
\begin{pmatrix}
x_1x_2 - y_1y_2 - z_1z_2 - t_1t_2 & x_1y_2 + y_1x_2 + z_1t_2 - t_1y_2 & x_1t_2 + y_1z_2 - t_1z_2 & x_1y_2 + y_1x_2 + z_1t_2 + t_1y_2 \\
-x_1y_2 + y_1x_2 + z_1t_2 - t_1y_2 & x_1y_2 + y_1x_2 + z_1t_2 + t_1y_2 & x_1t_2 - y_1z_2 + t_1z_2 & x_1y_2 + y_1x_2 + z_1t_2 + t_1y_2 \\
(x_1z_2 - y_1y_2 - z_1z_2 + t_1t_2) & -(x_1z_2 - y_1y_2 - z_1z_2 + t_1t_2) & x_1z_2 + y_1x_2 + z_1t_2 - t_1y_2 & x_1z_2 - y_1y_2 + z_1z_2 - t_1t_2 \\
-(x_1z_2 + y_1x_2 + z_1t_2 - t_1t_2) & -(x_1z_2 - y_1y_2 - z_1z_2 + t_1t_2) & x_1z_2 - y_1y_2 - z_1z_2 + t_1t_2 & x_1z_2 + y_1x_2 + z_1t_2 - t_1t_2
\end{pmatrix}
\]

So the bijection preserves the multiplicative structure. Moreover the bijection sends identity to identity. Therefore, since \( S^3 \) is group, \( S \) is also a group. Moreover this bijection is a homeomorphism because these two spaces are metrizable and the bijection just multiplies or divides the distance by 2.

**Notations**

For \( A \in \text{Mat}_n(\mathbb{C}) \) we write \( \overline{A} \) for the complex conjugate matrix of \( A \). Let \( n \in \mathbb{N} \). We write \( U_n := \{ A \in \text{Mat}_n(\mathbb{C}) \mid A^T A = I_n \} \). We write \( SU_n := \{ A \in U_n \mid \det(A) = 1 \} \).

**Remark 3.48**

If \( A \in U_n \) there exists \( \theta \in [0, 2\pi[ \) such that \( \det(A) = e^{i\theta} \) because the determinant of a matrix in \( U_n \) has its complex norm equal to 1.

**Remarks 3.49**

The group \( GL_n(\mathbb{C}) \) is a topological group with is open in \( \text{Mat}_n(\mathbb{C}) \) so locally compact by Example 2.12.

The topological group \( U_n(\mathbb{C}) \) is closed in \( GL_n(\mathbb{C}) \) because it is the inverse image of \( \{ I_n \} \) by the continuous map \( GL_n(\mathbb{C}) \rightarrow GL_n(\mathbb{C}), A \rightarrow A^T A \). So this group is locally compact.

Similarly \( SU_n(\mathbb{C}) \) is closed in \( U_n(\mathbb{C}) \) and so closed in \( GL_n(\mathbb{C}) \), so locally compact.
Example 3.50
We want to study $U_2$. Let us write

$$ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_2; \text{ we will deduce conditions on } a, b, c, d. \text{ We have} $$

$$ A \overline{\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi & \tau \\ \overline{\pi} & \overline{\tau} \end{pmatrix} = \begin{pmatrix} a \pi + b \overline{\pi} & a \tau + b \overline{\tau} \\ c \pi + d \overline{\pi} & c \tau + d \overline{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} $$

To summarize and using Remark 3.48, we have,

$$ \pi c + \overline{\pi} d = 0 \quad a \pi + b \overline{\pi} = \pi + d \overline{\pi} = 1 \quad -bc + ad = e^{i\theta}, \text{ for } a, b \in \mathbb{R}. $$

Seen as a system of equations in $c$ and $d$, we can write

$$ \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \begin{pmatrix} \pi \\ \overline{\pi} \end{pmatrix} = \begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix}. $$

So,

$$ \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} \begin{pmatrix} \pi \\ \overline{\pi} \end{pmatrix} = \begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix} = \begin{pmatrix} -\overline{b} e^{i\theta} \\ \pi e^{i\theta} \end{pmatrix}. $$

So we can write

$$ A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, $$

with $\theta \in \mathbb{R}$ and $a, b \in \mathbb{C}$ such that $a \pi + b \overline{\pi} = 1$.

If we take $\theta = 0$ then $A \in SU_2$. So we can write

$$ SU_2 = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \text{Mat}_2(\mathbb{C}) \mid a \pi + b \overline{\pi} = 1 \right\}. $$

We claim $SU_2$ is isomorphic (as topological group) to $S^3$ via the following bijection:

$$ \begin{pmatrix} x + iy \\ -z + it \\ x - iy \end{pmatrix} \leftrightarrow x + iy + jz + kt, $$

for $x, y, z, t \in \mathbb{R}$. Indeed it is clearly a bijection, continuous in both directions, so a homeomorphism. To check that it is a homomorphism, let us write the multiplication of two elements in $SU_2$. We write $a = x_1 + iy_1$, $b = z_1 + it_1$, $a' = x_2 + iy_2$ and $b' = z_2 + it_2$, for $x_1, y_1, z_1, t_1, x_2, y_2, z_2, t_2 \in \mathbb{R}$.

$$ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} a' & b' \\ -\overline{b'} & \overline{a'} \end{pmatrix} = \begin{pmatrix} a' a & b' a \\ a' b & b' b \end{pmatrix} $$

$$ = \begin{pmatrix} (x_1 + iy_1)(x_2 + iy_2) - (z_1 + it_1)(z_2 + it_2) & (x_1 + iy_1)(z_2 + it_2) + (z_1 + it_1)(x_2 - iy_2) \\ -(x_1 - iy_1)(z_2 - it_2) - (z_1 - it_1)(x_2 + iy_2) & (x_1 - iy_1)(x_2 - iy_2) - (z_1 - it_1)(z_2 + it_2) \end{pmatrix} $$

$$ = \begin{pmatrix} x_1 x_2 - y_1 y_2 - z_1 z_2 - t_1 t_2 + i(x_1 y_2 + y_1 x_2 + z_1 t_2 - t_1 z_2) & x_1 z_2 - y_1 t_2 + z_1 x_2 + t_1 y_2 + i(x_1 t_2 + y_1 z_2 - z_1 y_2 + t_1 x_2) \\ -x_1 z_2 + y_1 t_2 - z_1 x_2 - t_1 y_2 + i(x_1 t_2 + y_1 z_2 - z_1 y_2 + t_1 x_2) & x_1 x_2 - y_1 y_2 - z_1 z_2 - t_1 t_2 - i(x_1 y_2 + y_1 x_2 + z_1 t_2 - t_1 z_2) \end{pmatrix}. $$

This is exactly the form required. So the bijection preserves the multiplication.

### 3.4 Uniform structure of a topological group

The reader has probably seen that the axioms TG I and TG II for the filters of neighborhoods of the identity resemble to the axioms of a uniform structure. This is not a coincidence as we will see.

Let $(G, \tau)$ be a topological group with identity $e$ and let $\mathcal{B}$ be the filter of neighborhoods of $e$.

**Notation**

For $V \in \mathcal{B}$, we write $\tilde{V}_1 := \{(x, y) \in G \times G \mid x^{-1} y \in V \}$ and $\tilde{V}_r := \{(x, y) \in G \times G \mid y x^{-1} \in V \}$.

We are using the notations from [Bon60] §3.

We adopt the notations from [Mer10] 3.1 for the composition and the inversion of entourages.
Proposition 3.53

For every \( g \in G \), the left multiplication \( g \mapsto g'g \) and the right multiplication \( g \mapsto gg' \) are uniformly continuous isomorphisms \( (G, \mathcal{U}) \to (G, \mathcal{U}) \).

Proof. Let \( \tilde{V}_1 \in \mathcal{U} \) for some \( V \in \mathcal{B} \).

1. Left multiplication: if \( (g_1, g_2) \in \tilde{V} \) (i.e. \( g_1^{-1}g_2 \in V \)), then \( (g_1'g_2) \) is in \( \tilde{V} \) because \( (g_1'g_2) \) is uniformly continuous. Changing \( g' \) by \( g^{-1} \), we have that the inverse is also uniformly continuous.

2. Right multiplication: Let \( V' \in \mathcal{B} \) such that \( g'^{-1}V'g' \in V \). If \( (g_1, g_2) \in \tilde{V} \), then \( (g_1'g_2) \in \tilde{V} \). Indeed \( (g_1'g_2) \) is uniformly continuous. Changing \( g' \) by \( g^{-1} \), we have that the inverse is also uniformly continuous.

By an analogous argument, we have;

Proposition 3.54

For every \( g' \in G \), the left multiplication \( g \mapsto g'g \) and the right multiplication \( g \mapsto gg' \) are uniformly continuous isomorphism \( (G, \mathcal{U}) \to (G, \mathcal{U}) \).

Proposition 3.55

The symmetry \( g \mapsto g^{-1} \) is a uniformly continuous bijection \( (G, \mathcal{U}) \to (G, \mathcal{U}) \).

Proof. Let \( \tilde{V}_1 \in \mathcal{U} \) for some \( V \in \mathcal{B} \). Let \( V' \in \mathcal{B} \) such that \( g'^{-1}V'g' \in V \). If \( (g_1, g_2) \in \tilde{V} \), then \( (g_1^{-1}g_2) \in \tilde{V} \). Indeed \( g_1^{-1}g_2 \in V^{-1} \). Let \( \tilde{V}_2 \in \mathcal{U} \) for some \( V \in \mathcal{B} \). Let \( V' \in \mathcal{B} \) such that \( V'^{-1}V' \in V \). If \( (g_1, g_2) \in \tilde{V} \), then \( (g_1^{-1}, g_2^{-1}) \in \tilde{V} \).

One must be careful: in general, the symmetry is not uniformly continuous \( (G, \mathcal{U}) \to (G, \mathcal{U}) \) and \( (G, \mathcal{U}) \to (G, \mathcal{U}) \). And the multiplication \( G \times G \to G \) is in general not uniformly continuous. But we have the following Lemma.

We keep the notation \( \text{Topolun} \) for the functor \( \text{Unif} \to \text{Top} \) defined in [Mer10] 3.2.

Lemma 3.56

The topology induced by the left uniformity and the topology induced by the right uniformity are the same and are the initial topology \( \tau \). i.e. \( \text{Topolun}(G, \mathcal{U}) = \text{Topolun}(G, \mathcal{U}) = (G, \tau) \)

Proof. Let us recall that we have:

\[ \tau = \{ O \subseteq G \mid \forall x \in G, \exists V \in \mathcal{B} \text{ such that } xV \subseteq O \} = \{ O \subseteq G \mid \forall x \in G, \exists V \in \mathcal{B} \text{ such that } Vx \subseteq O \} \]

We have

\[ \tau_{\mathcal{U}} = \{ O \subseteq G \mid \forall x \in G, \exists V \in \mathcal{U} \text{ such that } xV \subseteq O \} = \{ O \subseteq G \mid \forall x \in G, \exists V \in \mathcal{U} \text{ such that } Vx \subseteq O \} \]

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This proves that the multiplication is uniformly continuous.

Lemma 3.57
If the uniform structures $\mathcal{U}_1$ and $\mathcal{U}_r$ are the same then the multiplication $G \times G \to G$ and the inversion $G \to G$ are uniformly continuous maps.

Proof. We already know by Proposition 3.55 that the inversion is uniformly continuous. Let us prove that the multiplication is uniformly continuous. We first prove that for every $V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that for every $g \in G$, $gWg^{-1} \subset W$. The important fact is that $W$ does not depend on $g$. Since $\mathcal{U}_1 = \mathcal{U}_r$, there exists $W \in \mathcal{B}$ such that $W_1 \subset V_r$. This means that if $x, y \in G$ are such that $x^{-1}y \in W$, then $yx^{-1} \in V$. Let $g \in G$ and let $z = gwg^{-1}$ for some $w \in W$. So we have $g^{-1}zg = w \in W$. Therefore, for every $g \in G$, $gWg^{-1} \subset V$.

Now let $U_1 \in \mathcal{U}_1$ such that $U_1U' \subset U$. Let $U'' \in \mathcal{B}$ be such that $gU''g^{-1} \subset U'$ for every $g \in G$. Now consider $(x_1, y_1, (x_2, y_2)) \in U''_1 \times U_1$, with our abuse of notation meaning $(x_1, x_2) \in U''_1$ and $(y_1, y_2) \in U_1$. So we have $x_1^{-1}x_2 \in U''$ and $y_2y_1^{-1} \in U'$.

In this case we have $(x_1y_1, x_2y_2) \in \tilde{U}_1$. Indeed,

$$(x_1y_1)^{-1}(x_2y_2) = y_1^{-1}x_1^{-1}x_2y_2 \in y_1^{-1}U''y_2 = y_1^{-1}U''y_1y_2^{-1}y_2 \subset U'y_1y_2^{-1}y_2 \subset U'U'' \subset U.$$

This proves that the multiplication is uniformly continuous.

Remark 3.58
The hypothesis of Lemma 3.57 is satisfied in particular if $G$ is abelian. Using Th. 1 p.27 in TG II [Bou71], one deduces that the hypothesis of Lemma 3.57 is also satisfied if $G$ is compact.

Proposition 3.59
Let $G, G'$ be two topological groups such that the right and the left uniform structures coincide and let $f : G \to G'$ be a continuous homomorphism. Then $f$ is uniformly continuous.

Proof. Let $\tilde{V}_1$ be an entourage in $G'$ with $V \in \mathcal{B}$. Since $f$ is continuous, there exists $U \in \mathcal{B}$ such that $g \in U \implies f(g) \in V$. The set $\tilde{U}_1$ is an entourage in $G$ and satisfies: $(x_1, x_2) \in \tilde{U}_1 \implies (f(x_1), f(x_2)) \in \tilde{V}_1$. Indeed let $(x_1, x_2) \in \tilde{U}_1$, i.e. $x_1^{-1}x_2 \in U$. We have

$$f(x_1)^{-1}f(x_2) = f(x_1^{-1})f(x_2) = f(x_1^{-1}x_2) \in V,$$

so $(f(x_1), f(x_2)) \in \tilde{V}_1$.

3.5 Group of homomorphisms
Let $G_1$ be a group and $G_2$ be an abelian group. The set $\text{Grp}(G_1, G_2)$ of homomorphisms $G_1 \to G_2$ can be equipped with a group structure as follows. For $f_1, f_2$, we define $f_1 \cdot f_2$ by $f_1 \cdot f_2(x) := f_1(x)f_2(x)$. Using the fact that $G_2$ is abelian, one checks that $f_1 \cdot f_2$ is indeed a homomorphism $G_1 \to G_2$. On the other hand one check easily that this $\cdot$ defines a group structure on $\text{Grp}(G_1, G_2)$. For $f \in \text{Grp}(G_1, G_2)$ we write $f^{-1}$ for the inverse of $f$ for the law $\cdot$. So $f^{-1}(x) = (f(x))^{-1}$. Moreover the group $\text{Grp}(G_1, G_2)$ is abelian. Now assume that $(G_1, \tau_1)$ is a topological group and $(G_2, \tau_2)$ is an abelian topological group. We consider the set $\text{Unif-grp}(G_1, G_2)$ to be the set of uniformly continuous homomorphisms $G_1 \to G_2$. By uniformly continuous, one means for both the left and right uniform structures; (see subsection 3.4).

Proposition 3.60
If $f, f_1, f_2 \in \text{Unif-grp}(G_1, G_2)$ then $f^{-1} \in \text{Unif-grp}(G_1, G_2)$ and $f_1 \cdot f_2 \in \text{Unif-grp}(G_1, G_2)$.

Proof. We already know that they are homomorphisms. The map $f^{-1}$ is the composition $\text{inv} \circ f$, where $\text{inv}$ is the inversion in $G_2$ which is uniformly continuous by Lemma 3.57. So $f^{-1}$ is uniformly continuous as a composition of uniformly continuous functions (see Proposition 3.4 in [Mer10]). Similarly, $f_1 \cdot f_2$ is the composition $\prod (f_1 \times f_2)$, where $\prod$ means the product in $G_2$ which is uniformly continuous by Lemma 3.57. The map $f_1 \times f_2$ is uniformly continuous by Remark 2.16. So $f_1 \cdot f_2$ is uniformly continuous as a composition of uniformly continuous functions.
So, Unif-grp\((G_1,G_2)\) is a subgroup of Grp\((G_1,G_2)\). Now we will see that it is in fact a topological group. To see this we prove that the multiplication \(\cdot\) and the inversion are uniformly continuous for the sigma-convergence uniform structure (see subsection 2.1) (we see Unif-grp\((G_1,G_2)\) as a subset of Set\((G_1,G_2)\)). So, let \(\mathcal{E}\) be a family of subsets of \(G_1\).

**Lemma 3.61**

The maps

\[
\text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \times \text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \longrightarrow \text{Unif-grp}_{\mathcal{E}}(G_1,G_2)
\]

\[(f_1,f_2) \longmapsto f_1 \cdot f_2 \]

and

\[
\text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \longrightarrow \text{Unif-grp}_{\mathcal{E}}(G_1,G_2)
\]

\[f \longmapsto f^{-1} \]

are uniformly continuous.

**Proof.** The multiplication is the composition of the following maps.

\[
\text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \times \text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \longrightarrow \text{Unif-grp}_{\mathcal{E}}(G_1,G_2) \times G_2 \longrightarrow \text{Unif-grp}_{\mathcal{E}}(G_1,G_2).
\]

The first map is the canonical identification defined in Proposition 2.19 which is uniformly continuous. The second is the map defined in Proposition 2.20 with \(h\) being the multiplication \(G_1 \times G_1 \rightarrow G_1\). The second map is uniformly continuous since the multiplication \(G_2 \times G_2 \rightarrow G_2\) is (see Lemma 3.57). So, the multiplication \(\cdot\) is uniformly continuous as a composition of uniformly continuous maps.

Taking the inverse is the left composition in Proposition 2.20 with \(h\) being the inversion \(G_2 \rightarrow G_2\) which is uniformly continuous (see Lemma 3.57). So taking the inverse is uniformly continuous by Proposition 2.20. \(\Box\)

### 3.6 Hyperspace of a topological group

**Definition**

We call the category of **uniform monoids** the category for which the objects are the monoids endowed with a uniform structure such that the binary operation is uniformly continuous, and the morphisms are the uniformly continuous maps which preserve the binary operation.

We will only consider the topological groups such that the left and the right uniform structures (see subsection 3.4) coincide.

Let \(G\) be a topological groups such that the left and the right uniform structures coincide, for example \(G\) abelian or compact. We write \(\mathcal{U}\) for the uniform structure on \(G\), \(e\) for the identity, \(\mathcal{B}\) for the filter of neighborhoods of \(e\) and \(\Pi\) for the product \(G \times G \rightarrow G\). We will equip the set \(\mathcal{K} G\) of non empty compact subsets of \(G\) with a topological monoid structure. We adopt the notation in [Mer10] 4.3. We consider the uniform space \((\mathcal{K} G, \mathcal{U}^1)\) as defined in subsection 2.1. So a basis for the uniformity \(\mathcal{U}^1\) is given by the family of sets

\[
Q(V) := \{(A,B) \in \mathcal{K}(X) \times \mathcal{K}(X) \mid A \subset V[B] \text{ and } B \subset V[A]\},
\]

for \(V \in \mathcal{B}\), or equivalently

\[
Q(V_r) := \{(A,B) \in \mathcal{K}(X) \times \mathcal{K}(X) \mid A \subset V_r[B] \text{ and } B \subset V_r[A]\},
\]

for \(V \in \mathcal{B}\). Now, we define a binary operation \(\Pi^1 : \mathcal{K} G \times \mathcal{K} G \rightarrow \mathcal{K} G\) by \((A,B) \mapsto A \cdot B := \{ab \mid a \in A, b \in B\}\).

The fact that \(A \cdot B \in \mathcal{K} G\) follows from the fact that it is the image of a compact by a continuous map (the product). One shows easily that \(\Pi^1\) is associative and that the compact singleton \(\{e\}\) is the identity. Now we prove that this operation is in fact uniformly continuous.

**Theorem 3.62**

The map:

\[
\Pi^1 : (\mathcal{K} G, \mathcal{U}^1) \times (\mathcal{K} G, \mathcal{U}^1) \longrightarrow (\mathcal{K} G, \mathcal{U}^1)
\]

\[(A,B) \longmapsto A \cdot B \]

is uniformly continuous.
Proof. The map $\Pi^1$ can be written as the composition

\[
\begin{array}{cccc}
(\mathcal{X}G, \mathcal{U}^1) \times (\mathcal{X}G, \mathcal{U}^1) & \xrightarrow{\Pi} & (\mathcal{X}(G \times G), (\mathcal{U} \times \mathcal{U})^1) & \xrightarrow{\mathcal{Q}\Pi} & (\mathcal{X}G, \mathcal{U}^1).
\end{array}
\]

\[(A, B) \mapsto A \times B \mapsto A \cdot B.\]

The first map is the Cartesian product which is uniformly continuous by Theorem 2.21. The second map is the image of $\Pi$ by the functor $\mathcal{Q}$ (see [Mer10] 4.3) which is uniformly continuous by [Mer10] Lemma 4.28. So, $\Pi^1$ is uniformly continuous as a composition of uniformly continuous maps.

This proves that $((\mathcal{X}G, \mathcal{U}^1), \Pi)$ is a uniform monoid, so a topological monoid. Now, let us consider another topological group $G'$ such that the left and the right uniform structures coincide. In [Mer10] Lemma 4.28, it is proved that if $f : G \to G'$ is uniformly continuous, then $\mathcal{Q}f : (\mathcal{X}G, \mathcal{U}^1) \to (\mathcal{X}G', \mathcal{U}'^1)$ is uniformly continuous. On the other hand, if $f$ is a homomorphism, then $\mathcal{Q}f$ satisfies $\mathcal{Q}f(A \cdot B) = \mathcal{Q}f(A) \cdot \mathcal{Q}f(B)$ for all $A, B \in \mathcal{X}G$. Indeed, we have

\[
\mathcal{Q}f(A \cdot B) = \mathcal{Q}f([ab \mid a \in A, b \in B]) = \{f(ab) \mid a \in A, b \in B\}
\]

\[
= \{f(a)f(b) \mid a \in A, b \in B\} = \{xy \mid x \in \mathcal{Q}f(A), y \in \mathcal{Q}f(B)\} = \mathcal{Q}f(A) \cdot \mathcal{Q}f(B).
\]

These last results prove that we have a functor $\mathcal{Q}_{Mon}$ from the full-subcategory of topological groups for which the two uniform structures coincide, to the category of uniform monoids. We also have proved that the functor $\mathcal{Q}_{Mon}$ restricts and co-restricts to the category of uniform monoids.
References


