Nilpotent centralizers and Springer isomorphisms

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\textbf{A B S T R A C T}

Let $G$ be a semisimple algebraic group over a field $K$ whose characteristic is very good for $G$, and let $\sigma$ be any $G$-equivariant isomorphism from the nilpotent variety to the unipotent variety; the map $\sigma$ is known as a Springer isomorphism. Let $y \in G(K)$, let $Y \in \text{Lie}(G)(K)$, and write $C^G = C^G(y)$ and $C^G = C^G(Y)$ for the centralizers. We show that the center of $C^G$ and the center of $C^G$ are smooth group schemes over $K$. The existence of a Springer isomorphism is used to treat the crucial cases where $y$ is unipotent and where $Y$ is nilpotent.

Now suppose $G$ to be quasisplit, and write $C$ for the centralizer of a rational regular nilpotent element. We obtain a description of the normalizer $N_G(C)$ of $C$, and we show that the automorphism of $\text{Lie}(C)$ determined by the differential of $\sigma$ at zero is a scalar multiple of the identity; these results verify observations of J.-P. Serre.

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\section{1. Introduction}

Let $G$ be a reductive group over the field $K$ and suppose $G$ to be $D$-standard; this condition means that $G$ satisfies some standard hypotheses which will be described in Section 3.2. For now, note that a semisimple group $G$ is $D$-standard if and only if the characteristic of $K$ is very good for $G$.

Consider the closed subvariety $N$ of nilpotent elements of the Lie algebra $g = \text{Lie}(G)$ of $G$, and the closed subvariety $U$ of unipotent elements of $G$. Since $G$ is $D$-standard, one may follow the argument given by Springer and Steinberg \cite{1,3.12} to find a $G$-equivariant isomorphism of varieties $\sigma : N \rightarrow U$. The mapping $\sigma$ is called a Springer isomorphism. There are many such maps: the Springer isomorphisms can be viewed as the points of an affine variety whose dimension is equal to the semisimple rank of $G$; see the note of Serre found in \cite[Appendix]{2} which shows that despite the abundance of such maps, each Springer isomorphism induces the same bijection between the (finite) sets of $G$-orbits in $N$ and in $U$. For some more details, see Section 3.3.

Let $y \in G(K)$ and $Y \in g(K)$. Since $G$ is $D$-standard, we observe in (3.4.1) -- following Springer and Steinberg \cite{1} -- that the centralizers $C^G(y)$ and $C^G(Y)$ are smooth group schemes over $K$. The first main result of this paper is as follows:

\textbf{Theorem A.} Let $Z_y = Z(C^G(y))$ and $Z_Y = Z(C^G(Y))$ be the centers of the centralizers.

(a) $Z_y$ and $Z_Y$ are smooth group schemes over $K$.
(b) $Y \in \text{Lie}(Z_Y)$.

See Section 2.6 for more details regarding the subgroup schemes $Z_y \subset C^G(y)$ and $Z_Y \subset C^G(Y)$. The existence of a Springer isomorphism plays a crucial role in the proof of \textbf{Theorem A}. 

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Keep the assumptions on $G$, and suppose in addition that $G$ is quasisplit over $K$; under these assumptions, one can find a $K$-rational regular nilpotent element $X \in \mathfrak{g}(K)$ [2, Theorem 54]. Write $C = C_0(X)$ for the centralizer of $X$; it is a smooth group scheme over $K$ (3.4.1).

Our next result concerns the normalizer of $C$ in $G$; write $N = N_G(C)$.

**Theorem B.** (i) $N$ is smooth over $K$ and is a solvable group.
(ii) If $r$ denotes the semisimple rank of $G$, then $\dim N = 2r + \dim \zeta_G$, where $\zeta_G$ denotes the center of $G$.
(iii) There is a 1 dimensional torus $S \subset N$ which is not central in $G$ such that $S \cdot \zeta_G$ is a maximal torus of $N$.

Fix now a cocharacter $\phi$ associated with the nilpotent element $X$; cf. (5.2.1).

**Theorem C.** Assume that the derived group of $G$ is quasisimple. Then the Lie algebra of $N/C$ decomposes as the direct sum

$$\text{Lie}(N/C) = \text{Lie}(S_0) \oplus \bigoplus_{i=2}^r \text{Lie}(N/C)(\phi; 2k_i - 2),$$

where $k_1 \leq k_2 \leq k_3 \leq \cdots \leq k_r$ are the exponents of the Weyl group of $G$, and where $S_0$ is the image of $S$ in $N/C$.

We will deduce several consequences from Theorems B and C. The first of these is:

**Theorem D.** The unipotent radical of $N/\text{Kalg}$ arises by base change from a split unipotent $K$-subgroup of $N$.

In older language, Theorem D asserts that the unipotent radical of $N$ is defined and split over $K$. Next, fix a Springer isomorphism $\sigma$ and write $u = \sigma(X)$. The unipotent radical of the group $C$ is defined over $K$, and $C$ is the product of $R_0(C)$ with the center $\zeta_G$ of $G$; see (5.2.4). The restriction of $\sigma$ to $R_0(C)$ yields an isomorphism of varieties

$$\gamma = \sigma|_{\text{Lie}(R_0(C))} : \text{Lie}(R_0(C)) \sim R_0 C$$

satisfying $\gamma(0) = \sigma|_{\text{Lie}(R_0(C))}(0) = 1$. So the tangent mapping $d\gamma_0$ yields a linear automorphism of the tangent space

$$\text{Lie}(R_0 C) = T_1 (R_0 C).$$

**Theorem E.** Suppose that the derived group of $G$ is quasisimple.

(1) The mapping $(d\gamma)_0$ is a scalar multiple of the identity automorphism of $\text{Lie}(R_0(C))$.
(2) Let $B$ a Borel subgroup of $G$ with unipotent radical $U$. Then $\sigma|_{\text{Lie}U} : \text{Lie} U \to U$ is an isomorphism, and $(d(\sigma|_{\text{Lie}U})_0 : \text{Lie} U \to \text{Lie} U$ is a scalar multiple of the identity.

We remark that Theorems B, C and E confirm the observations made by Serre at the end of [2, Appendix].

The paper is organized as follows. In Section 2 we recall some generalities about group schemes and smoothness; in particular, we describe conditions under which the center of a smooth group scheme is itself smooth. In Section 3 we recall some facts about reductive groups that we require; in particular, we define $D$-standard groups and we recall that element centralizers in $D$-standard groups are well-behaved. In Section 4 we give the proof of Theorem A. Finally, Section 5 contains the proofs of Theorems B–E.

## 2. Recollections: Group schemes

The main objects of study in this paper are group schemes over a field $K$. For the most part, we restrict our attention to affine group schemes $A$ of finite type over $K$. We begin with some general definitions.

### 2.1. Basic definitions

We collect here some basic notions and definitions concerning group schemes; for a full treatment, the reader is referred to [3] or to [4, part I].

For a commutative ring $\Lambda$, let us write $\text{Alg}_\Lambda$ for the category of “all” commutative $\Lambda$-algebras. $^1$ We will write $\Lambda' \in \text{Alg}_\Lambda$ to mean that $\Lambda'$ is an object of this category—i.e. that $\Lambda'$ is a commutative $\Lambda$-algebra.

We are going to consider affine schemes over $\Lambda$; an affine scheme $X$ is determined by a commutative $\Lambda$-algebra $R$: the algebra $R$ determines a functor $X : \text{Alg}_\Lambda \to \text{Sets}$ by the rule

$$X(\Lambda') = \text{Hom}(R, \Lambda').$$

The scheme $X$ is “$\Lambda$” this functor, and one says that $X$ is represented by the algebra $R$. One usually writes $R = \Lambda[X]$ and one says that $\Lambda[X]$ is the coordinate ring of $X$. The affine scheme $X$ has finite type over $\Lambda$ provided that $\Lambda[X]$ is a finitely generated $\Lambda$-algebra.

A group valued functor $A$ on $\text{Alg}_\Lambda$ which is an affine scheme will be called an affine group scheme. If $A$ is an affine group scheme, then $\Lambda[A]$ has the structure of a Hopf algebra over $\Lambda$.

If $\Lambda' \in \text{Alg}_\Lambda$, we write $A_{\Lambda/\Lambda'}$ for the group scheme over $\Lambda'$ obtained by base change. Thus $A_{\Lambda/\Lambda'}$ is the group scheme over $\Lambda'$ represented by the $\Lambda'$-algebra $\Lambda[A] \otimes_{\Lambda} \Lambda'$.

$^1$ Taken in some universe, to avoid logical problems.
Let us fix an affine group scheme $A$ of finite type over the field $K$. Write $K[A]$ for the coordinate algebra of $K$, and choose an algebraic closure $K_{\text{alg}}$ of $K$.

2.2. Comparison with algebraic groups

In many cases, the group schemes we consider may be identified with a corresponding algebraic group; we now describe this identification.

If the algebra $K[A]$ is geometrically reduced – i.e. is such that $K_{\text{alg}}[A] = K[A] \otimes_K K_{\text{alg}}$ has no non-zero nilpotent elements – then also $K[A]$ is reduced. The $K_{\text{alg}}$-points $A(K_{\text{alg}})$ of $A$ may be viewed as an affine variety over $K_{\text{alg}}$; since it is reduced, $K_{\text{alg}}[A]$ is the algebra of regular functions on $A(K_{\text{alg}})$. Moreover, $A(K_{\text{alg}})$ together with the $K$-algebra $K[A]$ of regular functions on $A(K_{\text{alg}})$ may be viewed as a variety defined over $K$ in the sense of [5] or [6].


The constructions in the preceding paragraphs are inverse to one another, and these constructions permit us to identify the category of linear algebraic groups defined over $K$ with the full subcategory of the category of affine group schemes of finite type over $K$ consisting of those group schemes with geometrically reduced coordinate algebras.

There are interesting group schemes in characteristic $p > 0$ whose coordinate algebras are not reduced. Standard examples of non-reduced group schemes include the group scheme $\mu_p$ represented by $K[T]/(T^p - 1)$ with co-multiplication given by $\Delta(T) = T \otimes T$, and the group scheme $\alpha_p$, represented by $K[T]/(T^p)$ with co-multiplication given by $\Delta(T) = T \otimes I + 1 \otimes T$. Note that $\mu_p$ is a subgroup scheme of the multiplicative group $G_m$, and $\alpha_p$ is a subgroup scheme of the additive group $G_0$.

2.3. Smoothness

For $A \in \text{Alg}_{K}$, let $A[\epsilon]$ denote the algebra of dual numbers over $A$; thus $A[\epsilon]$ is a free $A$-module of rank 2 with $A$-basis $\{1, \epsilon\}$, and $\epsilon^2 = 0$. If $A$ is a group scheme over $K$, the natural $A$-algebra homomorphisms

$$A \leftrightarrow A[\epsilon] \xrightarrow{\pi} A$$

yield corresponding group homomorphisms

$$A(A) \leftrightarrow A(A[\epsilon]) \xrightarrow{A(\pi)} A(A).$$

The Lie algebra $\text{Lie}(A)$ of $A$ is the group functor on $\text{Alg}_{K}$ given for $A \in \text{Alg}_{K}$ by

$$\text{Lie}(A)(A) = \ker(A(A[\epsilon]) \xrightarrow{A(\pi)} A(A)).$$

Abusing notation somewhat, we are going to write also $\text{Lie}(A)$ for $\text{Lie}(A)(K)$. We have:

(2.3.1) ([3, II.4]).

(a) $\text{Lie}(A)$ has the structure of a $K$-vector space, and the mapping $\text{Lie}(A) \rightarrow \text{Lie}(A)(A)$ induces an isomorphism

$$\text{Lie}(A)(A) \simeq \text{Lie}(A) \otimes_K A$$

for each $A \in \text{Alg}_{K}$.

(b) For $A \in \text{Alg}_{K}$ and $g \in A(A)$, the inner automorphism $\text{Int}(g)$ determines by restriction a $A$-linear automorphism $\text{Ad}(g)$ of $\text{Lie}(A)(A) \simeq \text{Lie}(A) \otimes_K A$; thus $\text{Ad} : A \rightarrow \text{GL}(\text{Lie}(A))$ is a homomorphism of group schemes over $K$.

(2.3.2) ([3, II.5.2.1, p. 238] or [7, (21.8) and (21.9)]). One says that the group scheme $A$ is smooth over $K$ if any of the following equivalent conditions holds:

(a) $A$ is geometrically reduced—i.e. $A_{K_{\text{alg}}}$ is reduced.

(b) the local ring $K[A]_I$ is regular, where $I$ is the maximal ideal defining the identity element of $A$.

(c) the local ring $K[A]_I$ is regular for each prime ideal $I$ of $K[A]$.

(d) $\dim_K \text{Lie}(A) = \dim A$, where $\dim A$ denotes the dimension of the scheme $A$, which is equal to the Krull dimension of the ring $K[A]$.

If $A$ is a group scheme over $K$, we often abbreviate the phrase “$A$ is smooth over $K$” to “$A$ is smooth”;

2.4. Reduced subgroup schemes

The following result is well-known; a proof may be found in [8, Lemma 3].

(2.4.1). If $K$ is perfect, there is a unique smooth subgroup $A_{\text{red}} \subset A$ which has the same underlying topological space as $A$. If $B$ is any smooth group scheme over $K$ and $f : B \rightarrow A$ is a morphism, then $f$ factors in a unique way as a morphism $B \rightarrow A_{\text{red}}$ followed by the inclusion $A_{\text{red}} \rightarrow A$.

Note that if $K$ is not perfect, the subgroup scheme $(A/I_{K_{\text{alg}}})_{\text{red}}$ of $A_{K_{\text{alg}}}$ may not arise by base change from a subgroup scheme over $K$; see [8, Example 4].
2.5. Fixed points and the center of a group scheme

For the remainder of Section 2, let us fix a group scheme $A$ which is affine and of finite type over the field $K$. Let $V$ denote an affine $K$-scheme (of finite type) on which $A$ acts. Define a $K$-subfunctor $W$ of $V$ as follows: for each $A \in \text{Alg}_K$, let

$$W(A) = \{ v \in V(A) \mid av = v \text{ for each } a \in A \}$$

We write $W = V^A$; it is the functor of fixed points for the action of $A$.

In general one indeed must define the set $W(A)$ as the fixed point set of all $a \in A(A')$ for varying $A'$; e.g. if $A$ is infinitesimal, $A(K) = \{1\}$ while $W(K)$ is typically a proper subset of $V(K)$.

Since $V$ is affine – hence separated – and since $K$ is a field so that $K[A]$ is free over $K$, we have:

(2.5.1) ([3, II.1 Theorem 3.6] or [4, I.2.6.10]). $V^A$ is a closed subscheme of $V$.

The following assertion is somewhat related to [4, I.2.7 (11) and (12)].

(2.5.2). Suppose in addition that $A$ is smooth over $K$. Then for any commutative $K$-algebra $K'$ which is an algebraically closed field, we have $V^A(K') = V^{A(K')}_K$.

Proof. It is immediate from definitions that $V^{A(K')} = V(K')^{A(K')}$ and $V^{A(K')}_K$. In order to prove the inclusion $V^{A(K')} \subseteq V^{A(K')}_K$, we will assume (for notational convenience) that $K = K'$ algebraically closed. Suppose that $v \in V(K)$ and that $v$ is fixed by each element of $A(K)$.

Consider now the morphism $\phi : A \to V$ given for each $A \in \text{Alg}_K$ and each $a \in A(A)$ by the rule $a \mapsto av$. The result will follow if we argue that $\phi$ is a constant morphism. But we know that $\phi : A(K) \to V(K)$ is constant. Since $A$ is a reduced scheme, the morphism $\phi$ is determined by its values on closed points; since $K$ is algebraically closed, the closed points are in bijection with $A(K)$; the fact that $\phi$ is constant now follows. □

Consider now the action of $A$ on itself by inner automorphisms. For any $A \in \text{Alg}_K$ and any $a \in A(A)$, let us write $\text{Int}(a)$ for the inner automorphism $x \mapsto axa^{-1}$ of the $A$-group scheme $A_{/A}$. The fixed point subscheme for this action is by definition the center $Z$ of $A$; thus we have the following result (see also [3, II.1.3.9]):

(2.5.3). The center $Z$ is a closed subgroup scheme of $A$. For any $A \in \text{Alg}_K$, we have

$$Z(A) = \{ a \in A(A) \mid \text{Int}(a) \text{ is the trivial automorphism of the group scheme } A_{/A} \}$$

2.6. Smoothness of the center

Write $a = \text{Lie}(A)$ for the Lie algebra of $A$. Recall from (2.3.1) the adjoint action $\text{Ad}$ of $A$ on $a$.

(2.6.1). Regarding $a$ as a $K$-scheme, the Lie algebra of $Z$ is the fixed point subscheme of $a$ for the adjoint action of $A$.

Proof. Since $Z$ is the fixed point subscheme of $A$ for the action of $A$ on itself by inner automorphisms, the assertion follows from [3, II.4.2.5]. □

In particular, $\text{Lie}(Z)$ identifies with the $K$-points $a^{\text{Ad}(A)}(K)$ of this fixed point functor, and one recovers the fixed point functor from the $K$-points [4, I.2.10(3)]:

$$a^{\text{Ad}(A)}(A) = \text{Lie}(Z) \otimes_K A.$$

(2.6.2). The center $Z$ of $A$ is smooth over $K$ if and only if

$$\dim Z = \dim_K a^{\text{Ad}(A)}(K) = \dim_K \text{Lie}(Z).$$

Proof. Immediate from (2.3.2) and the observation (2.6.1). □

Example. Let $K$ be a perfect field of characteristic $p > 0$, and let $A$ be the smooth group scheme over $K$ for which

$$A(A) = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^p & s \\ 0 & 0 & 1 \end{pmatrix} \mid t \in A^\times, s \in A \right\}$$

for each $A \in \text{Alg}_K$. The Lie algebra $a$ is spanned as a $K$-vector space by the matrices

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$
We are going to argue that \( Z \) is not smooth—i.e., \( Z \neq Z_{\text{red}} \). Observe first that \( a \) is an Abelian Lie algebra; thus its center \( z(a) \) is all of \( a \).

Now, if \( K_{\text{alg}} \) is an algebraic closure of \( K \), it is easy to check that the center of the group \( A(K_{\text{alg}}) \) is trivial. It follows that the smooth group scheme \( Z_{\text{red}} \) satisfies \( Z_{\text{red}}(K_{\text{alg}}) = 1 \); thus \( Z_{\text{red}} \) is trivial and \( \text{Lie}(Z_{\text{red}}) = 0 \).

It is straightforward to verify that the multiples of \( X \) are the only fixed points of \( a \) under the adjoint action of \( A \). Thus \( \text{Lie}(Z) = a_{\text{Ad}(A)} \) has dimension 1 as a \( K \)-vector space. Since \( \dim Z = \dim Z_{\text{red}} = 0 \), it follows that \( Z \) is not smooth.

Note that for this example, both containments in the following sequence are proper:

\[
\text{Lie}(Z) \subset \text{Lie}(Z_{\text{red}}) \subset z(a).
\]

2.7. Smoothness of certain fixed point subgroup schemes

Recall that a group scheme \( D \) over \( K \) is diagonalizable if \( K[D] \) is spanned as a linear space by the group of characters \( X^*(D) \). The group scheme \( D \) is of multiplicative type if \( D/{\text{alg}(K)} \) is diagonalizable.

Suppose in this section that \( D \) is either a group scheme of multiplicative type, or that \( D \) is an étale group scheme over \( K \) for which the finite group \( D/{\text{alg}(K)} \) has order invertible in \( K \).

Assume that \( D \) acts on the group scheme \( A \) by group automorphisms: for any \( A \in \text{Alg}_K \) and any \( x \in D(A) \), the element \( x \) acts on the group scheme \( A / \Lambda \) as a group scheme automorphism.

The fixed points \( A^D \) form a closed subgroup scheme of \( A \). Moreover, we have:

(2.7.1). If \( A \) is smooth over \( K \), then also the fixed point subgroup scheme \( A^D \) is smooth over \( K \).

**Proof.** According to the “Théorème de lisité des centralisateurs” [3, Il.5.2.8 (p. 240)] the result will follow if we know that \( H^1(D, \text{Lie}(A)) = 0 \). It suffices to check this condition after extending scalars; thus we may and will suppose that \( D \) is diagonalizable or that \( D \) is the constant group scheme determined by a finite group whose order is invertible in \( K \).

In each case, one knows that the cohomology group \( H^n(D, M) \) is 0 for all \( D \)-modules \( M \) and all \( n \geq 1 \); for a finite group with order invertible in \( K \), this vanishing is well-known; for a diagonalizable group, see [4, I.4.3]. \( \square \)

2.8. Possibly disconnected groups

Let \( G \) be a smooth linear algebraic group over \( K \).

(2.8.1). Suppose that \( 1 \to G \to G_1 \to E \to 1 \) is an exact sequence, where \( E \) is finite étale and \( E({\text{alg}(K)}) \) has order invertible in \( K \). If the center of \( G \) is smooth, then the center of \( G_1 \) is smooth.

**Proof.** Write \( Z \) for the center of \( G \). Write \( Z_1 \) for the center of \( G_1 \). Note that \( E \) acts naturally on \( Z \).

There is an exact sequence of groups

\[
1 \to Z^E \to Z_1 \to H \to 1
\]

for a subgroup \( H \subset E \). Since \( Z \) is smooth, the smoothness of \( Z^E \) follows from (2.7.1); since \( H \) is smooth, one obtains the smoothness of \( Z_1 \) by applying [7, Cor. (22.12)] \( \square \).

2.9. Split unipotent radicals

Fix a smooth group scheme \( A \) over \( K \). A smooth group scheme \( B \) over \( K \) is unipotent if each element of \( B({\text{alg}(K)}) \) is unipotent. Recall that the unipotent radical of \( A / {\text{alg}(K)} \) is the maximal closed, connected, smooth, normal, unipotent subgroup scheme of \( A / {\text{alg}(K)} \).

(2.9.1) ([6, Prop. 14.4.5]). If \( K \) is perfect, there is a smooth subgroup scheme \( R_uA \subset A \) such that \( R_uA / {\text{alg}(K)} \) is the unipotent radical of \( A / {\text{alg}(K)} \).

If \( K \) is not perfect, then in general \( R_uA / {\text{alg}(K)} \) does not arise by base change from a \( K \)-subgroup scheme of \( A \). The unipotent group \( B \) is said to be split provided that there are closed subgroup schemes

\[
1 = B_0 \subset B_1 \subset \cdots \subset B_n = B
\]

such that \( B_i/B_{i-1} \cong G_i \) for \( 1 \leq i \leq n \).

**Theorem.** Let \( A \) be a connected, solvable, and smooth group scheme over \( K \). Let \( T \subset A \) be a maximal torus, and suppose that \( \phi : G_m \to T \) is a cocharacter. Write \( S \) for the image of \( \phi \). If \( \text{Lie}(T) \) is precisely the set of fixed points \( \text{Lie}(A)^S \), and if each non-zero weight \( \lambda \) of \( S \) on \( \text{Lie}(A) \) satisfies \( \langle \lambda, \phi \rangle > 0 \), then \( R_uA \) is defined over \( K \) and is a split unipotent group scheme.

**Proof.** Write \( P = P(\phi) \) for the smooth subgroup scheme of \( A \) determined by \( \phi \) as in [6, Section 13.4]; it is the subgroup contracted by the cocharacter \( \phi \). Write \( M = C_A(S) \); \( M \) is connected [6, p. 110] and smooth [3, p. 476, cor. 2.5]. There is a smooth, connected, normal, unipotent subgroup scheme \( U(\phi) \subset P \) for which the product morphism

\[
M \times U(\phi) \to P
\]
is an isomorphism of varieties; [6, 13.4.2]. Moreover, since \( (\lambda, \phi) > 0 \) for each weight of \( S \) on \( \text{Lie}(A) \), it follows that \( U(\phi) \) is trivial. Thus loc. cit. 13.4.4 shows that \( A = P \).

Evidently \( T \subseteq M \). Since \( \text{Lie}(T) = \text{Lie}(M) \), it follows that \( M = T \). It follows that \( U(\phi)/K_{\text{alg}} \) is the unipotent radical of \( A/K_{\text{alg}} \) as desired.

Finally, it follows from [6, 14.4.2] that \( U(\phi) \) is a \( K \)-split unipotent group, and the proof is complete. □

2.10. Torus actions on a projective space

Let \( T \) be a split torus over \( K \), and let \( V \) be a \( T \)-representation. For \( \lambda \in X^*(T) \), let \( V_\lambda \) be the corresponding weight space; thus \( T \) acts on \( V_\lambda \) through the character \( \lambda : T \to G_m \). There are distinct characters \( \lambda_1, \ldots, \lambda_n \in X^*(T) \) such that

\[
V = \bigoplus_{i=1}^n V_{\lambda_i},
\]

the \( \lambda_i \) are the weights of \( T \) on \( V \). Let us fix a vector \( 0 \neq v \in V_{\lambda_1} \).

Consider now the projective space \( \mathbf{P}(V) \) of lines through the origin in \( V \); for a non-zero vector \( w \in V \), write \([w]\) for the corresponding point of \( \mathbf{P}(V) \). The linear action of \( T \) on \( V \) induces in a natural way an action of \( T \) on \( \mathbf{P}(V) \).

Since \( v \) is a weight vector for \( T \), the point \([v]\) is fixed by the action of \( T \). Consider the tangent space \( M = T_{[v]}\mathbf{P}(V) \); since \([v]\) is a fixed point of \( T \), the action of \( T \) on \( \mathbf{P}(V) \) determines a linear representation of \( T \) on \( M \).

(2.10.1). The non-zero weights of \( T \) on \( M = T_{[v]}\mathbf{P}(V) \) are the characters \( \lambda_i - \lambda_1 \) for \( 1 < i \leq n \). Moreover,

\[
\dim M_0 = \dim V_{\lambda_1} - 1 \quad \text{and} \quad \dim M_{\lambda_i - \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.
\]

Proof. Choose a basis \( S_1, S_2, \ldots, S_n \) for the dual space of \( V^\vee \) for which \( S_i \in V_{\lambda_i} \) for \( 1 \leq i \leq r \) i.e. the vector \( S_i \) has weight \( -\lambda_i \) for the contragredient action of \( T \) on \( V^\vee \). Without loss of generality, we may and will assume that \( S_1 \) satisfies \( S_1(v) \neq 0 \) and that \( S_i(v) = 0 \) for \( 2 \leq i \leq n \).

Now consider the affine open subset \( \mathcal{V} = \mathbf{P}(V)_{S_1} \) of \( \mathbf{P}(V) \) defined by the non-vanishing of \( S_1 \). One knows that \([v]\) is a point of \( \mathcal{V} \). Moreover, \( \mathcal{V} \simeq \text{Aff}^{r-1} \) where \( r = \dim V \). Since \( S_1 \) is a weight vector for the action of the torus \( T \), it is clear that \( \mathcal{V} \) is a \( T \)-stable subvariety of \( \mathbf{P}(V) \). More precisely, \( \mathcal{V} \) identifies with the affine scheme \( \text{Spec}(A) \) where \( A \) is the \( T \)-stable subalgebra

\[
A = k \left[ \frac{S_2}{S_1}, \frac{S_3}{S_1}, \ldots, \frac{S_n}{S_1} \right]
\]

of the field of rational functions \( k(\mathbf{P}(V)) \).

Under this identification, the point \([v]\) of \( \mathcal{V} \) corresponds to the point \( \bar{v} \) of \( \text{Aff}^{r-1} \); i.e. to the maximal ideal \( m = (\frac{S_2}{S_1}, \frac{S_3}{S_1}, \ldots, \frac{S_n}{S_1}) \subset A \). Now, \( m \) and \( m^2 \) are \( T \)-invariant; since \( \frac{S_i}{S_1} \) has weight \( -\lambda_i + \lambda_1 \), evidently the weights of \( T \) in its representation on \( m/m^2 \) are of the form \( -\lambda_i + \lambda_1 \), and one has

\[
\dim(m/m^2)_0 = \dim V_{\lambda_1} - 1 \quad \text{and} \quad \dim(m/m^2)_{-\lambda_i + \lambda_1} = \dim V_{\lambda_i}, \quad 1 < i \leq n.
\]

The assertion now follows since there is a \( T \)-equivariant isomorphism between the tangent space to \( \mathbf{P}(V) \) at \([v]\) i.e. the space \( M = T_{[v]}\mathbf{P}(V) \), and the contragredient representation \( (m/m^2)^\vee \). □

2.11. Surjective homomorphisms between group schemes: Normalizers

In this section, let us fix group schemes \( G_1 \) and \( G_2 \) over \( K \), and suppose that \( f : G_1 \to G_2 \) is a surjective homomorphism of group schemes; recall that \( f \) is surjective provided that the comorphism \( f^* : K[G_2] \to K[G_1] \) is injective (cf. [7, Prop. 22.3]).

The mapping \( f \) is said to be separable provided that \( df : \text{Lie}(G_1) \to \text{Lie}(G_2) \) is surjective as well.

Let \( C_2 \subseteq G_2 \) be a subgroup scheme, and let \( C_1 = f^{-1}C_2 \) be the scheme-theoretic inverse image.

(2.11.1). (a) The mapping obtained by restriction \( f_{|C_1} : C_1 \to C_2 \) is surjective.
(b) If \( C_1 \) is smooth, then \( C_2 \) is smooth.
(c) If \( f \) is separable and \( C_2 \) is smooth, then \( C_1 \) is smooth.
(d) Suppose that \( f \) is separable, and that either \( C_1 \) or \( C_2 \) is smooth. Then both \( C_1 \) and \( C_2 \) are smooth, and \( f_{|C_1} \) is separable.

Proof. (a) and (b) follow from [7, Prop. 22.4].

We now prove (c). Since \( f \) is separable and surjective, [7, Prop. 22.13] shows that \( \ker f \) is a smooth group scheme over \( K \). Note that \( \ker f \subseteq C_1 \). If \( C_2 \) is smooth, the smoothness of \( C_1 \) now follows from [7, Cor. 22.12].

We finally prove (d). The smoothness assertions have already been proved. We again know \( \ker f \) to be smooth over \( K \). In particular, \( \dim \ker f = \dim df \). Since \( \ker f \subseteq C_1 \), we have

\[
\dim \text{image}(df_{|C_1}) = \dim \text{Lie}(C_1) - \dim \ker df_{|C_1} = \dim C_1 - \dim \ker df_{|C_1} = \dim C_2,
\]

where we have used [7, Prop. 22.11] for the final equality; since \( C_2 \) is smooth, it follows that \( df_{|C_1} : \text{Lie}(C_1) \to \text{Lie}(C_2) \) is surjective. □
Write \( N_2 = N_{G_2}(C_2) \) for the normalizer of \( C_2 \) in \( G_2 \). Thus \( N_2 \) is the subgroup functor given for \( A \in \text{Alg}_K \) by the rule
\[
N_2(A) = \{ g \in G_2(A) \mid g \text{ normalizes the subgroup scheme } C_{2/A} \subset G_{2/A} \} = \{ g \in G_2(A) \mid gC_2(A')g^{-1} = C_2(A') \text{ for all } A' \in \text{Alg}_A \}.
\]
According to [3, II.1 Theorem 3.6 (b)], \( N_2 \) is a closed subgroup scheme of \( G_2 \).
As a consequence of (2.11.1), we find the following:

(2.11.2). Set \( N_1 = f^{-1}N_2 \).
(a) \( N_1 = N_{G_1}(C_1) \).
(b) \( f_{|N_1} : N_1 \to N_2 \) is surjective.
(c) If \( N_1 \) is smooth, then \( N_2 \) is smooth.
(d) If \( f \) is separable and \( N_2 \) is smooth, then \( N_1 \) is smooth.
(e) Suppose that \( f \) is separable and that either \( N_1 \) or \( N_2 \) is smooth. Then both \( N_1 \) and \( N_2 \) are smooth, and \( f_{|N_1} \) is separable.

3. Recollections: Reductive groups

Let \( G \) be a connected and reductive group over \( K \). Thus \( G \) is a smooth group scheme over \( K \), or equivalently \( G \) is a linear algebraic group defined over \( K \). To say that \( G \) is reductive means that the unipotent radical of \( G/K_{\text{alg}} \) is trivial. We are going to write \( \xi_G = Z(G) \) for the center of \( G \).

Some results will be seen to hold for a reductive group \( G \) in case \( G \) is \( D \)-standard; in the next few sections, we explain this condition. We must first recall the notions of good and bad characteristic.

3.1. Good and very good primes

Suppose that \( H \) is a smooth group scheme over \( K \) – i.e. an algebraic group over \( K \) – for which \( H/K_{\text{alg}} \) is quasisimple; thus \( H \) is geometrically quasisimple. Write \( R \) for the root system of \( H \). The characteristic \( p \) of \( K \) is said to be a bad prime for \( R \) – equivalently, for \( H \) – in the following circumstances: \( p = 2 \) is bad whenever \( R \neq A_r \), \( p = 3 \) is bad if \( R = G_2, F_4, E_7 \), and \( p = 5 \) is bad if \( R = E_8 \). Otherwise, \( p \) is good.

A good prime \( p \) is very good provided that either \( R \) is not of type \( A_r \), or that \( R = A_r \) and \( r \not\equiv -1 \pmod{p} \).

If \( H \) is any reductive group, one may apply [7, Theorem 26.7 and 26.8]\(^3\) to see that there is a possibly inseparable central isogeny
\[
R(H) \times \prod_{i=1}^{m} H_i \to H
\]
where the radical \( R(H) \) of \( H \) is a torus, and where for \( 1 \leq i \leq m \) there is an isomorphism \( H_i \simeq R_i/K_i \) for a finite separable field extension \( L_i/K \) and a geometrically quasisimple, simply connected group scheme \( J_i \) over \( L_i \); here, \( R_i/K_i \) denotes the “Weil restriction” – or restriction of scalars – of \( J_i \) to \( K \); cf. [6, Section 11.4]. The \( H_i \) are uniquely determined by \( H \) up to order of the factors. Then \( p \) is good, respectively very good, for \( H \) if and only if that is so for \( J_i \) for every \( 1 \leq i \leq m \).

3.2. \( D \)-standard

Recall from Section 2.7 the notion of a diagonalizable group scheme, and of a group scheme of multiplicative type.

(3.2.1). If \( D \) is subgroup scheme of \( G \) of multiplicative type, the connected centralizer \( C_{c}(D)^o \) is reductive.

When \( D \) is smooth, the preceding result is well-known: the group \( D \) is the direct product of a torus and a finite étale group scheme all of whose geometric points have order invertible in \( K \). The centralizer of a torus is (connected and) reductive, and one is left to apply a result of Steinberg [9, Cor. 9.3] which asserts that the centralizer of a semisimple automorphism of a reductive group has reductive identity component. In fact, the result remains valid when \( D \) is no longer smooth; a proof will appear elsewhere.

Consider reductive groups \( H \) which are direct products
\[
(\ast) \quad H = H_1 \times T
\]
where \( T \) is a torus, and where \( H_1 \) is a semisimple group for which the characteristic of \( K \) is very good.

Definition. A reductive group \( G \) is \( D \)-standard if there exists a reductive group \( H \) of the form \( (\ast) \), a subgroup \( D \subset H \) such that \( D \) is of multiplicative type, and a separable isogeny between \( G \) and the reductive group \( C_H(D)^o \)\(^4\).

(3.2.2) ([2, Remark 3]). For any \( n \geq 1 \), the group \( \text{GL}_n \) is \( D \)-standard. The group \( \text{SL}_n \) is \( D \)-standard if and only if \( p \) does not divide \( n \).

\(^3\) [7] only deals with the semisimple case; the extension to a general reductive group is not difficult to handle, and an argument is sketched in the footnote found in [8, Section 2.4].

\(^4\) This definition does not require the knowledge that \( C_{c}(D)^o \) is reductive: if there is an isogeny between \( G \) and \( C_{c}(D)^o \), then \( C_{c}(D)^o \) is reductive.
In order to prove (3.2.4) below, we first observe:

(3.2.3). Let $M, G_1, G_2$ be affine group schemes of finite type over $K$. Let $f : G_1 \to G_2$ be a surjective morphism of group schemes, suppose that $\ker f$ is central in $G_1$, and let $\phi : M \to G_2$ be a homomorphism of group schemes for which $\phi^{-1}(\zeta_{G_2})$ is central in $M$. Consider the group scheme $\tilde{M}$ defined by the Cartesian diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{G_1} & G_2 \\
\phi \downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{f} & \tilde{G_2}
\end{array}
\]

Then

(a) $\phi^{-1}(\zeta_{G_1})$ is central in $\tilde{M}$.

(b) Suppose that $G_1, G_2$ are connected and reductive, that $f$ is a separable isogeny, and that $M$ is connected and quasisimple. Then $\tilde{M}$ is connected and quasisimple.

**Proof.** To prove (a), let $N = \phi^{-1}(\zeta_{G_1})$. It is enough to show that $\tilde{\phi}(N)$ is central in $G_1$ and that $\tilde{f}(N)$ is central in $M$. The first of these observations is immediate from definitions, while the second follows from assumption on the mapping $\phi : M \to G_2$ once we observe that $f(N) \subseteq \phi^{-1}(\zeta_{G_2})$.

For (b), we view $f$ as arising by base change from $f$. Then $\tilde{f}$ is an isogeny since $\ker(f)_{\text{alg}}$ and $\ker(\tilde{f})_{\text{alg}}$ coincide. Moreover, it follows from [10, Prop 4.3.22] that $\tilde{f}$ is separable (since it is étale). Thus $\tilde{f}$ is a separable isogeny; since $\tilde{M}$ is separably isogenous to a connected quasisimple group, it is itself connected and quasisimple. \(\square\)

(3.2.4). Suppose that the $D$-standard reductive group $G$ is split over $K$. There are $D$-standard reductive groups $M_1, \ldots, M_d$ together with a homomorphism $\Phi : M \to G$, where $M = \prod_{i=1}^d M_i$, such that the following hold:

(a) The derived group of $M_i$, is geometrically quasisimple for $1 \leq i \leq d$.

(b) $\Phi$ is surjective and separable.

(c) For $1 \leq i < j \leq d$, the image in $G$ of $M_i$ and $M_j$ commute.

(d) The subgroup scheme $\Phi^{-1}(\zeta_G)$ is central in $\prod_{i=1}^d M_i$.

**Proof.** We argue first that it suffices to prove the result after replacing $G$ by a separably isogenous group. More precisely, we prove: (*) if $f : G_1 \to G_2$ is a separable isogeny between $D$-standard reductive groups $G_1$ and $G_2$, then (3.2.4) holds for $G_1$ if and only if it holds for $G_2$.

Suppose first that the conclusion of (3.2.4) is valid for $G_1$. If $\Phi : M \to G_1$ is a homomorphism for which (a)-(d) hold, then evidently (a)-(d) hold for $f \circ \Phi$.

Now suppose that the conclusion of (3.2.4) is valid for $G_2$, and that $\Phi : M \to G_2$ is a homomorphism for which (a)-(d) hold. For each $1 \leq j \leq d$ write $\Phi_j$ for the composite of $\Phi$ with the inclusion of $M_j$ in the product. Form the group $\tilde{M}_j = M_j \times_{G_1} G_1$ as in (3.2.3). Then by (b) of loc. cit., $\tilde{M}_j$ is quasisimple. Moreover, loc. cit. (a) shows the kernel of $\tilde{\Phi}_j$ to be central in $\tilde{M}_j$.

Note that the image of $\tilde{\Phi}_j$ is mapped to the image of $\phi_j$ by $f$. Now, $f$ is a separable isogeny, hence in particular $f$ is central; i.e. $\ker f$ is central. It follows that the image of $\tilde{\Phi}_j$ commutes with the image of $\tilde{\Phi}_l$ whenever $1 \leq i \neq j \leq n$. We can thus form the homomorphism $\tilde{\Phi} : \prod_{j=1}^d \tilde{M}_j \to G_1$ whose restriction to each $\tilde{M}_j$ is just $\tilde{\Phi}_j$, and it is clear that (a)-(d) hold for $\tilde{\Phi}$; this completes the proof of (*).

In view of the definition of a $D$-standard group, we may now suppose that $G$ is the connected centralizer $\text{C}_G(D)^{\nu}$ of a diagonalizable subgroup scheme $D \subset H_1 = H \times T$, where $H$ is a semisimple group in very good characteristic and $T$ a torus.

We may use [6, 8.1.5] to write $G$ as a commuting product of its minimal non-trivial connected, closed, normal subgroups $J_i$ for $i = 1, 2, \ldots, n$. Fix a maximal torus $T \subset G$, so that $T_i = (T \cap J_i)^{\nu}$ is a maximal torus of $J_i$ for each $i$.

Now set $T' = \prod_{i \neq J} T_i$; then $T'$ is a torus in $G$. Moreover, $J_i$ is the derived subgroup of the reductive group $M_i = \text{C}_G(T_i)$.

Now, $M_i$ is the connected centralizer in $H_1$ of the diagonalizable subgroup $(T', D)$; thus $M_i$ is $D$-standard. Finally, putting $M = \prod_i M_i$, we have a natural surjective mapping $M \to G$ for which (a)-(d) hold, as required. \(\square\)

3.3. Existence of Springer isomorphisms

Let $G$ denote a $D$-standard reductive group. We write $\mathcal{N} = \mathcal{N}(G) \subset g$ for the nilpotent variety of $G$ and $\mathcal{U} = \mathcal{U}(G) \subset G$ for the unipotent variety of $G$.

By a Springer isomorphism, we mean a map

$$\sigma : \mathcal{N} \to \mathcal{U}$$

which is a $G$-equivariant isomorphism of varieties over $K$.

The first assertion of the following Theorem – the existence of a Springer isomorphism – is due essentially to Springer; see e.g. [1, III.3.12] for the case of an algebraically closed field, or see [11]. The second assertion was obtained by Serre and appears in the appendix to [2].
Theorem. (Springer, Serre).

1. There is a Springer isomorphism \( \sigma : \mathcal{N} \to \mathcal{U} \).
2. Any two Springer isomorphisms induce the same mapping between the set of \( G(K_{\text{alg}}) \) – orbits in \( \mathcal{U}(K_{\text{alg}}) \) and the set of \( G(K_{\text{alg}}) \) – orbits in \( \mathcal{N}(K_{\text{alg}}) \), where \( K_{\text{alg}} \) is an algebraic closure of \( K \).

Proof. We sketch the argument for assertion (1) in order to point out the role of the \( D \)-standard assumption made on \( G \).

If \( G \) is semisimple in very good characteristic, the nilpotent variety \( \mathcal{N} \) and the unipotent variety \( \mathcal{U} \) are both normal. Indeed, for \( \mathcal{U} \), one knows [1, III.2.7] that \( \mathcal{U} \) is normal whenever \( G \) is simply connected (with no condition on \( p \)). Moreover, one knows that the normality of \( \mathcal{U} \) is preserved by separable isogeny.\(^5\) In positive characteristic the normality of \( \mathcal{N} \) for a semisimple group \( G \) is a result of Veldkamp (for most \( p \)) and of Demazure when the characteristic is very good for \( G \); see [12, 8.5]. Using the normality of \( \mathcal{U} \) and of \( \mathcal{N} \), Springer showed that [11] there is a \( G \)-equivariant isomorphism as required.

To conclude that assertion (1) is valid for any \( D \)-standard groups, it suffices to observe the following: (i) if \( \pi : G \to G_1 \) is a separable isogeny, then there is a Springer isomorphism for \( G \) if and only if there is a Springer isomorphism for \( G_1 \), and (ii) if \( H \) is a reductive group for which there is a Springer isomorphism, and if \( D \subset H \) is a subgroup of multiplicative type, then \( C_H^p(D) \) has a Springer isomorphism. \( \Box \)

We note a related result for certain not-necessarily-connected reductive groups.

(3.3.1). Let \( G \) be a connected reductive group for which there is a Springer isomorphism \( \sigma : \mathcal{N}(G) \to \mathcal{U}(G) \). Let \( D \subset G \) be a subgroup of multiplicative type, and let \( M = C_G(D) \).

(a) \( \sigma \) restricts to an isomorphism \( \mathcal{N}(M) \to \mathcal{U}(M) \).

(b) The finite group \( M(K_{\text{alg}})/M^o(K_{\text{alg}}) \) has order invertible in \( K \).

Proof. Assertion (a) follows from the observations: \( \mathcal{N}(M) = \mathcal{N}(G)^D \) and \( \mathcal{U}(M) = \mathcal{U}(G)^D \). To prove (b), note that \( \mathcal{N}(M) = \mathcal{N}(M^o) \) is connected, so that by (a), also \( \mathcal{U}(M) \) is connected. Thus \( \mathcal{U}(M) \subset M^o \) and (b) follows at once. \( \Box \)

3.4. Smoothness of some subgroups of \( D \)-standard groups

For any algebraic group, and any element \( x \in G \), let \( C_G(x) \) denote the centralizer subgroup scheme of \( G \). Then by definition \( \text{Lie} \, C_G(x) = c_G(x) \), where \( c_G(x) \) denotes the centralizer of \( x \) in the Lie algebra \( g \), but since the centralizer may not reduced, the dimension of \( c_G(x) \) may be larger than the dimension \( \dim c_G(x) = \dim c_G(x)_\text{red} \), where \( c_G(x)_\text{red} \) denotes the corresponding reduced – hence smooth – group scheme. Similar remarks hold when \( x \in G \) is replaced by an element \( X \in g \).

When \( G \) is a \( D \)-standard reductive group, this difficulty does not arise. Indeed:

(3.4.1). Let \( G \) be \( D \)-standard, let \( x \in G(K) \), and let \( X \in g = g(K) \). Then \( C_G(x) \) and \( C_G(X) \) are smooth over \( K \). In other words,

\[ \dim C_G(x) = \dim c_G(x) \quad \text{and} \quad \dim C_G(X) = \dim c_G(X). \]

In particular,

\[ \text{Lie} \, C_G(x)_\text{red} = c_G(x) \quad \text{and} \quad \text{Lie} \, C_G(X)_\text{red} = c_G(X). \]

Proof. When \( G \) is semisimple in very good characteristic, the result follows from [1, 1.5.2 and 1.5.6]. The extension to \( D \)-standard groups is immediate; the verification is left to the reader.\(^6\) \( \Box \)

Similar assertions holds for the center of \( G \), as follows:

(3.4.2). Let \( G \) be a \( D \)-standard reductive group. Then the center \( \zeta_G \) of \( G \) is smooth.

Proof. Indeed, for any field extension \( L \) of \( K \), the center of \( G \mid L \) is just the group scheme \( (\zeta_G) \mid L \) obtained by base change. To prove that \( \zeta_G \) is smooth, it suffices to prove that \( (\zeta_G) \mid L \) is smooth. So we may and will suppose that \( K \) is algebraically closed; in particular, \( G \) is split.

Fix a Borel subgroup \( B \) of \( G \) and fix a maximal torus \( T \subset B \). Let \( X = \sum_\alpha X_\alpha \in \text{Lie}(B) \) be the sum over the simple roots \( \alpha \), where \( X_\alpha \in \text{Lie}(B)_\alpha \) is a non-zero root vector; then \( X \) is regular nilpotent.

For a root \( \beta \in X^+(T) \) of \( T \) on \( \text{Lie}(G) \), write \( \beta^\vee \in X_\alpha(T) \) for the corresponding cocharacter \( \beta^\vee : G_m \to T \), and consider the cocharacter \( \phi : G_m \to T \) given by \( \phi = \sum_\beta \beta^\vee \in X_\alpha(T) \), where the sum is over all positive roots \( \beta \). Then \( \text{Ad}(\phi(t))X = t^2X \) for each \( t \in G_m(K) \) so that the image of \( \phi \) normalizes the centralizer \( C = C_G(X) \).

Now, \( C \) is a smooth subgroup of \( G \) by (3.4.1). The image of \( \phi \) is a torus, hence is a diagonalizable group. So the fixed points \( C^{\text{Im} \phi} \) of the image of \( \phi \) on \( C \) form a smooth subgroup by (2.7.1).

Finally, since \( X \) is contained in the dense \( B \)-orbit on \( \text{Lie}(R_B,B) \), \( X \) is a distinguished nilpotent element; cf. [12, 4.10, 4.13]. So it follows from [12, Prop. 5.10], that \( C^{\text{Im} \phi} \) is precisely \( \zeta_G \), the center of \( G \). Thus indeed \( \zeta_G \) is smooth. \( \Box \)

Remark. In case \( G \) is semisimple in very good characteristic one can instead apply [13, 0.13] to see that the center of the Lie algebra \( \text{Lie}(G) \) is trivial; this shows in this special case that \( \zeta_G \) is smooth.

\(^5\) More precisely, if \( \pi : G \to G_1 \) is a separable central isogeny, the restriction of \( \pi \) determines an isomorphism between \( \mathcal{U}(G) \) and \( \mathcal{U}(G_1) \).

\(^6\) Complete details of the reduction from the case of a \( D \)-standard group to that of a semisimple group in very good characteristic can be given along the lines of the argument used in the proof of (5.4.2).
3.5. The centralizer of a semisimple element of $\mathfrak{g}$

Suppose $G$ is $D$-standard, let $X \in \mathfrak{g} = \mathfrak{g}(K)$ be semisimple, and write $M = C_G(X)$. Recall that the closed subgroup scheme $M$ is smooth over $K$; cf. (3.4.1).

**(3.5.1).** (a) $X$ is tangent to a maximal torus $T$ of $G$.
(b) $M^\circ$ is a reductive group.

**Proof.** [5, Prop. 11.8 and Prop. 13.19].

Now fix a maximal torus $T$ with $X \in \text{Lie}(T)$ as in (3.5.1). Let us recall the following:

**(3.5.2).** If $S \subset G$ is a torus, there is a finite, separable field extension $L \supseteq K$ and a parabolic subgroup $P \subset G_{/L}$ such that $C_G(S)_{/L}$ is a Levi factor of $P$.

**Proof.** Let the finite separable field extension $L \supseteq K$ be a splitting field for $S$. The result then follows from [14, 4.15].

Suppose for the moment that the characteristic $p$ of $K$ is positive. Let $K_{\text{sep}}$ be a separable closure of $K$, and consider the (additive) subgroup $B$ of $K_{\text{sep}}$ generated by the elements $d\beta(X)$ for $\beta \in X^*(T/K_{\text{sep}})$; since $d\beta(X) = 0$ whenever $\beta \in pX^*(T/K_{\text{sep}})$, $B$ is a finite elementary Abelian $p$-group. Write $\Gamma' = \text{Gal}(K_{\text{sep}}/K)$ for the Galois group; since $X \in \mathfrak{g}(K)$, the group $B$ is stable under the action of $\Gamma'$.

Let $\mu = D(B)$ be the $K$-group scheme of multiplicative type determined by the $\Gamma'$-module $B$. The $\Gamma'$-equivariant mapping $X^*(T/K_{\text{sep}}) \rightarrow B$ given by $\beta \mapsto d\beta(X)$ determines an embedding of $\mu$ as a closed subgroup scheme of $T$.

**(3.5.3).** We have $M^\circ = C_{\mu}(\mu)^\circ$.

**Proof (Sketch).** Since $M^\circ$ and $C_{\mu}(\mu)^\circ$ are smooth groups over $K$, it suffices to give the proof after replacing $K$ by an algebraic closure. In that case $\mu$ is diagonalizable. Let $R \subset X^*(T)$ be the roots of $\mu$ for the torus $T$, and for $\alpha \in R$ let $U_{\alpha} \subset G$ be the corresponding root subgroup of $G$.

Then using the Bruhat decomposition of $G$, one finds that

$$M^\circ = \langle T, U_{\alpha} \mid \alpha(X) = 0 \rangle = C_{\mu}(\mu)^\circ;$$

the required argument is essentially the same as that given in [1, II.4.1] except that loc. cit. does not treat infinitesimal subgroup schemes; cf. [15] for the details.

**Theorem.** There is a finite separable field extension $L \supseteq K$ for which the connected centralizer $M_{/L}^\circ = C_{\mu}(\mu)^\circ_{/L}$ is a Levi factor of a parabolic subgroup of $G_{/L}$.

**Proof.** Suppose first that $K$ has characteristic $p > 0$. In view of (3.5.3), the reductive group $M^\circ$ is $D$-standard, since $\mu$ is a group of multiplicative type. According to (3.4.2), the center $Z$ of $M^\circ$ is smooth. Let $S$ be a maximal torus of $Z$. We have evidently $M^\circ \subset C_{\mu}(S)$. It follows that $\text{Lie}(Z) = \text{Lie}(S)$. We may now use (2.6.1) to see that $X \in \text{Lie}(Z) = \text{Lie}(S)$. Thus $M^\circ \subset C_{\mu}(S)$.

It follows that $M^\circ = C_{\mu}(S)$, and we conclude via (3.5.2).

The situation when $K$ has characteristic zero is simpler. In that case, the center $Z$ of the reductive group $M^\circ$ is automatically smooth. If $S$ is a maximal torus of $Z$ then $M^\circ = C_{\mu}(S)$ as before.

3.6. Borel subalgebras

Suppose that $K$ is algebraically closed. By a Borel subalgebra of $\mathfrak{g}$, we mean the Lie algebra $\mathfrak{b} = \text{Lie}(B)$ of a Borel subgroup $B \subset G$.

**Proposition (15, 14.25).** $\mathfrak{g}$ is the union of its Borel subalgebras. More precisely, for each $X \in \mathfrak{g}$, there is a Borel subalgebra $\mathfrak{b}$ with $X \in \mathfrak{b}$.

4. The center of a centralizer

For a $D$-standard reductive group $G$ over $K$, let $x \in G(K)$ and $X \in \mathfrak{g}(K)$. We are going to consider the centralizers $C_G(X)$ and $C_{\mu}(x)$, and in particular, the centers $Z_x = Z(C_G(X))$ and $Z_x = Z(C_{\mu}(x))$ of these centralizers. As we have seen, $Z_x$ is a closed subscheme of $C_G(x)$ and $Z_x$ is a closed subscheme of $C_{\mu}(x)$. In this section, we will prove Theorem A from the introduction; namely, in Section 4.2, we prove that $Z_x$ and $Z_x$ are smooth. In Section 4.1, we establish some preliminary results under the assumption that $K$ is perfect. Since the smoothness of $Z_x$ and of $Z_x$ will follow if it is proved after base change with an algebraic closure $K_{\text{alg}}$ of $K$, this assumption on $K$ is harmless for our needs.
4.1. Unipotence of the center of the centralizer when $X$ is nilpotent

Suppose in this section that the field $K$ is perfect; thus if $A$ is a group scheme over $K$, we may speak of the reduced subgroup scheme $A_{\text{red}}$—cf. (2.4.1). We begin with the following observation which is due independently to R. Proud and G. Seitz. For completeness, we include a proof.

(4.1.1) Let $x$ be unipotent, let $X$ be nilpotent, write $C$ for one of the groups $C_C(x)$ or $C_C(X)$, and write $Z = Z(C)$; thus $Z$ is one of the groups $Z_x$ or $Z_X$.

(a) $C^o$ is not contained in a Levi factor of a proper parabolic subgroup of $G$.

(b) The quotient $(Z_{\text{red}})^o/(\zeta G)^o$ is a unipotent group, where $Z_{\text{red}}$ is the corresponding reduced group, and $(Z_{\text{red}})^o$ is its identity component.

(c) Let $Y \in \text{Lie}(Z)$ be semisimple. Then $Y \in \text{Lie}(\zeta G)$.

Proof. It suffices to prove each of the assertions after extending scalars; thus, we may and will suppose in the proof that $K$ is algebraically closed. Moreover, if $\sigma : N \to U$ is a Springer isomorphism, then $C_C(X) = C_C(\sigma(X))$. Thus it suffices to give the proof for the centralizer of $X$.

We first prove (a). Suppose that $L$ is a Levi factor of a parabolic subgroup $P$, and assume that $C^o$ is a subgroup scheme of $L$. Then $C^o = C^o(C) = C^o_l(X)$ so that $L = \text{Lie} C = \text{Lie} C_l(X)$. Since $L$ is again a $D$-standard reductive group, we see by the smoothness of centralizers that $C_l(X)$ is the centralizer in $L$ of $X$ (3.4.1); in particular, it follows that every fixed point of $\text{ad}(X)$ on $\text{Lie}(G)$ lies in $\text{Lie}(L)$. If $L$ were a proper subgroup of $G$, the nilpotent operator $\text{ad}(X)$ would have a non-zero fixed point on $\text{Lie} P_0$, it follows that $L = G$.

We will now deduce (b) and (c) from (a). For (b), let $S \subset Z$ be a torus. Assertion (b) will follow if we prove that $S$ is central in $G$. But $L = C_l(S)$ is a Levi factor of some parabolic subgroup $P$ of $G$ by (3.5.2), and $C^o \subset L$. Thus by (a) we have $P = G = L$; this shows that $S$ is central in $G$, as required.

For (c), let $Y \in \text{Lie}(Z)$ be semisimple. According to Theorem 3.5, $L = C^o_l(Y)$ is a Levi factor of some parabolic subgroup $P$, and $C^o \subset L$. So again (a) shows that $P = G = L$. Since $C_C(Y) = G$, it follows that $Y$ is a fixed point for the adjoint action of $G$ on $\text{Lie}(G)$. But according to (2.6.1), we have $\text{Lie}(\zeta G) = \text{Lie}(G)^{\text{Ad}(G)}$; thus indeed $Y \in \text{Lie}(\zeta G)$ as required. □

As a consequence, we deduce the following structural results:

(4.1.2) With notation and assumptions as in (4.1.1), we have:

(a) $Z_{\text{red}}$ is the internal direct product $\zeta_G \cdot R_{\text{red}} Z_{\text{red}}$.

(b) The set of nilpotent elements of $\text{Lie}(Z)$ forms a subalgebra $u$ for which $\text{Lie} Z = \text{Lie}(\zeta G) \oplus u$.

Proof. Note that $Z$ and also $\text{Lie}(Z)$ are commutative; since the product of two commuting unipotent elements of $G$ is unipotent and the sum of two commuting nilpotent elements of $\text{Lie}(G)$ is nilpotent, results (a) and (b) follow from (4.1.1)(b) and (c). □

4.2. Smoothness of the center of the centralizer

In this section, $K$ is again arbitrary. Let $x \in G(K), X \in g(K)$ be arbitrary, write $C$ for one of the groups $C_C(x)$ or $C_C(X)$, and write $Z = Z(C)$, so that $Z$ is one of the groups $Z_x$ or $Z_X$. We are now ready to prove the following:

Theorem. The center $Z = Z(C)$ is a smooth group scheme over $K$.

Proof. Since a group scheme is smooth over $K$ if and only if it is smooth upon scalar extension, we may and will suppose $K$ to be algebraically closed (hence in particular perfect). So as in Section 4.1, we may speak of the reduced subgroup scheme $A_{\text{red}}$ of a group scheme $A$ over $K$.

Let $x = x_s x_u$ and $X = X_s + X_u$ be the Jordan decompositions of the elements; thus $x_s \in G$ and $X_s \in g$ are semisimple, $x_u \in G$ is unipotent, $X_u \in g$ is nilpotent, and we have: $x_s x_u = x_u x_s$ and $[X_s, X_u] = 0$.

Then

$$C_C(x) = C_M(x_u) \quad \text{and} \quad C_C(X) = C_M(X_u)$$

where $M = C_C(X_u)$ resp. $M = C_C(x_u)$.

Now, the Zariski closure of the group generated by $x_s$ is a smooth diagonalizable group whose centralizer coincides with $C_C(x_u)$. And according to Section 3.5 the centralizer of $X_s$ is reductive and is the centralizer of a (non-smooth) diagonalizable group scheme. Thus in both cases, the connected component of $M$ is itself a $D$-standard reductive group.

Moreover, (3.3.1) shows that $x_u$ is a $K$-point of $M^o$. There is an exact sequence

$$1 \to C_M(x_u) \to C_M(x_u) \to E \to 1$$

resp.

$$1 \to C_M(x_u) \to C_M(x_u) \to E' \to 1$$
for a suitable subgroup $E$ resp. $E'$ of $M/M^0$. Since $M/M^0$ has order invertible in $K$ (3.3.1), apply (2.8.1) to see that the smoothness of $Z$ follows from the smoothness of the center of $C_{M'}(x_0)$ resp. $C_{M'}(x_0)$; thus the proof of the theorem is reduced to the case where $x$ is unipotent and $X$ is nilpotent. Since in that case $C_C(X) = C_C(\sigma(X))$ where $\sigma : \mathcal{N} \to \mathcal{U}$ is a Springer isomorphism, we only discuss the centralizer of a nilpotent element $X \in \mathfrak{g}$.

We must argue that $\dim Z = \dim \text{Lie} Z$. Since it is a general fact that $\dim \text{Lie} Z \leq \dim Z$, it suffices to show the following:

(*): $\dim \text{Lie} Z \leq \dim Z$.

By (4.1.2) we have $\text{Lie} Z = \text{Lie}(\zeta_C) \oplus u$ where $u$ is the set of all nilpotent $Y \in \text{Lie} Z$. According to (3.4.2), the center $\zeta_C$ of $G$ is smooth. Thus $\dim \zeta_C = \dim \text{Lie} \zeta_C$. In view of (4.1.2), the assertion (*) will follow if we prove that

(**): $\dim u \leq \dim R_z\text{red}$.

In order to prove (**), we fix a Springer isomorphism $\sigma : \mathcal{N} \to \mathcal{U}$ – see Theorem 3.3 – and we consider the restriction of $\sigma$ to $u$.

We first argue that $\sigma$ maps $u$ to $R_z\text{red}$. Since $u$ is smooth – hence reduced – and since $K$ is algebraically closed, it suffices to show that $\sigma$ maps the $K$-points of $u$ to $R_z\text{red}$. Fix $Y \in u(K)$.

If $g \in \text{C}_C(X)(K)$, the inner automorphism $\text{Int}(g)$ of $C$ is trivial on $Z$; thus, the automorphism $\text{Ad}(g)$ of Lie $C$ is trivial on $\text{Lie} Z$. It follows that

$$g\sigma(Y)g^{-1} = \sigma(\text{Ad}(g)Y) = \sigma(Y).$$

Since $K$ is algebraically closed, it now follows from (2.5.2) that

$$\sigma(Y) \in Z(K) = \text{C}_C(X)\text{Int}(\text{C}_C(X))(K).$$

Since $u$ is reduced, one knows $\sigma(Y) \in Z_{\text{red}}(K)$. Since $\sigma(Y)$ is unipotent, it follows that $\sigma(Y) \in R_z\text{red}(K)$. Thus the restriction of the Springer isomorphism $\sigma$ gives a morphism $\sigma_u : u \to R_z\text{red}$. Since $\sigma$ is a closed morphism, it follows that the image of $\sigma_u$ is a closed subvariety of $R_z\text{red}$ whose dimension is $\dim u$, so that indeed (**) holds.

With notation as in the preceding proof, we point out a slightly different argument. Namely, reasoning as above, one can show that the inverse isomorphism $\tau = \sigma^{-1} : \mathcal{U} \to \mathcal{N}$ maps $R_z\text{red}$ to $u$. It follows that $R_z\text{red}$ and $u$ are isomorphic varieties, hence they have the same dimension.

Note that we have now proved Theorem A from the introduction.

5. Regular nilpotent elements

In this section, we are going to prove Theorems B, C and E from the introduction. We denote by $G$ a $D$-standard reductive group over the field $K$. Let $T \subset G$ be a maximal torus, and let $T_0 \subset T$ where $T_0$ is a maximal torus of the derived group $G' = (G, G)$ of $G$. Let us write $r = \dim T_0$ for the semisimple rank of $G$. Finally, let $W = N_G(T)/T \simeq N_{G'}(T_0)/T_0$ be the corresponding Weyl group.

5.1. Degrees and exponents

We give here a quick description of some well-known numerical invariants associated with the Weyl group $W$. We suppose that the derived group $G'$ of $G$ is quasisimple, and we suppose that $T$ (and hence $G$) is split over $K$.

Let $V = X^\times(T_0) \otimes \mathbb{Q}$ and note that the action of the Weyl group $W$ on $T_0$ determines a linear representation $(\rho, V)$ of $W$. The algebra of polynomials (regular functions) on $V$ may be graded by assigning the degree 1 to each element of the dual space $V^\vee \subset \mathbb{Q}[V]$. The action via $\rho$ of $W$ on $V$ determines an action of $W$ on $\mathbb{Q}[V]$ by algebra automorphisms, and it is known that the algebra $\mathbb{Q}[V]^W$ of $W$-invariant polynomials on $V$ is generated as a $\mathbb{Q}$-algebra by $r$ algebraically independent homogeneous elements of positive degree [16, V.5.3 Theorem 3]. The degrees of $W$ are the degrees $d_1, d_2, \ldots, d_r$ of a system of homogeneous generators for $\mathbb{Q}[V]^W$. The degrees depend – up to order – only on $W$; see [16, V.5.1]. The exponents of $W$ are the numbers $k_1, k_2, \ldots, k_r$, where $k_i = d_i - 1$ for $1 \leq i \leq r$.

Recall that the “exponents” earn their name as follows. Let $c \in W$ be a Coxeter element [16, V.6.1], and write $h$ for the order of $c$. If $E$ is a field of characteristic 0 containing a primitive $h$th root of unity $\sigma \in E^\times$, then [16, V.6.2 Prop. 3] the eigenvalues of $\rho(c)$ on $V \otimes \mathbb{Q} E$ are the values

$$\sigma^{k_1}, \sigma^{k_2}, \ldots, \sigma^{k_r}.$$

The exponents and degrees are known explicitly; cf. [16, Plate I-IX].

5.2. The centralizer of a regular nilpotent element

In this section, $G$ is again a $D$-standard reductive group (whose derived group is not required to be quasisimple) which we assume to be quasisplit over $K$.

If $\phi : G_m \to G$ is a cocharacter and $i \in \mathbb{Z}$, we write $\phi(i)$ for the $i$-weight space of the action of $\phi(G_m)$ on $\mathfrak{g}$ under the adjoint action of $\phi(G_m)$; thus

$$g(\phi; i) = \{Y \in \mathfrak{g} | \text{Ad}(\phi(t))Y = t^iY \forall t \in K_{\text{alg}}^\times\).$$
Any cocharacter $\phi$ determines a unique parabolic subgroup $P = P(\phi)$ whose $K_{\text{alg}}$-points are given by:

$$P(K_{\text{alg}}) = \left\{ g \in G(K_{\text{alg}}) \mid \lim_{t \to 0} \text{Int}(\phi(t))g \text{ exists} \right\}.$$ 

One knows that $p = \text{Lie}(P) = \sum_{i \geq 0} B(\phi; i)$.

Let $X \in \mathfrak{o}(K)$ be nilpotent. Following [12, Section 5.3], we say that a cocharacter $\psi : \mathbb{G}_m \to G$ is said to be associated to a nilpotent element $X$ in case (1) $X \in \mathfrak{o}(\psi; 2)$, and (ii) there is a maximal torus $S$ of the centralizer $C_S(X)$ such that the image of $\psi$ lies in $L$.

(5.2.1). (a) There are cocharacters associated to $X$.
(b) If $\phi$ and $\phi'$ are cocharacters associated to $X$, then $P(\phi) = P(\phi')$.
(c) The centralizer $C_S(X)$ is contained in $P = P(\phi)$ for a cocharacter $\phi$ associated to $X$.
(d) The unipotent radical $R$ of $C_S(X)/K_{\text{alg}}$ is defined over $K$ and is a $K$-split unipotent group.
(e) Any two cocharacters associated to $X$ are conjugate by a unique element of $R(K)$.

**Proof.** In the geometric setting, these assertions may be found in [12]; the existence of an associated cocharacter is an essential part of the Bala–Carter theorem, a conceptual proof of which may be found in [17]. Over the ground field $K$, (a) and (c) follow from [18, Theorem 26 and 28]. (b) follows since associated cocharacters are optimal for the unstable vector $X$ in the sense of Kempf; see [17]. Finally, (d) and (e) follow from [2, Prop/Defn 21]. \qed

Finally, recall that a nilpotent element $X \in \mathfrak{o}$ is distinguished provided that a maximal torus of the centralizer $C_S(X)$ is central in $G$.

(5.2.2). Let $X \in \mathfrak{o}$ be nilpotent. The following are equivalent:
(a) $X$ is regular—i.e. $\dim C_S(X)$ is equal to the rank of $G$.
(b) $X \in \text{Lie}(B)$ for precisely one Borel subgroup $B$ of $G$.

Moreover, if $X$ is regular then $X$ is distinguished, and if $\phi$ is a cocharacter associated with $X$, then $B = P(\phi)$ is the unique Borel subgroup with $X \in \text{Lie}(B)$.

**Proof.** The equivalence of (a) and (b) can be found in [12, Cor. 6.8]. Note that in loc. cit. it is assumed that $K$ is algebraically closed. But, it suffices to prove that (b) implies (a) after replacing $K$ by an extension field. It remains to argue that (a) implies (b). But given (a), one knows that there is a unique Borel subgroup $B \subset G/K_{\text{alg}}$ with $X \in \text{Lie}(B)$, where $K_{\text{alg}}$ is an algebraic closure of $K$. It now follows from [18, Prop. 27] that $B$ is a parabolic subgroup of $G$ [i.e. $B$ is defined over $K$], and (b) follows.

That a regular element is distinguished follows from the Bala–Carter theorem; it can be seen perhaps more directly just by observing that $B$ is a distinguished parabolic subgroup, so that an element of the dense orbit of $B$ on $\text{Lie}(R_0B)$ is distinguished by [19, 5.8.7].

Finally, write $P = P(\phi)$. It follows from [12, 5.9] that $X$ is in the dense $P$-orbit on $\text{Lie}(R_0P)$ and that $C_P(X) = C_S(X)$; thus $\dim \text{Ad}(G)X = 2\dim R_0P$ so that indeed $P$ must be a Borel subgroup. \qed

Since $G$ is assumed to be quasisplit, we have:

(5.2.3) ([2, Theorem 54]). There is a regular nilpotent element $X \in \mathfrak{o}(K)$.

We fix now a regular nilpotent element $X$. Let $C = C_S(X)$ be the centralizer of $X$, and write $\zeta_G$ for the center of $G$.

(5.2.4). For the group $C = C_S(X)$ we have:
(a) the maximal torus of $C$ is the identity component of the center $\zeta_G$ of $G$.
(b) $C = \zeta_G \cdot R_0(C)$.
(c) $C$ is commutative.

**Proof.** Assertions (a) and (b) follow from [12, Section 4.10, Section 4.13] precisely as in the proof of (3.4.2).

For (c), use a Springer isomorphism $\sigma : \mathcal{N} \to \mathcal{U}$, to see that $C$ is the centralizer of the regular unipotent element $u = \sigma(X)$. Then the commutativity of $C$ follows from a result of Springer – see [13, Theorem 1.14] – which implies that the centralizer of $u$ contains a commutative subgroup of dimension equal to the rank of $G$. This shows that the identity component of $C$ is commutative. Since $R_0C$ is connected and since $C = \zeta_G R_0C$, the group $C$ is itself commutative. \qed

We now fix a cocharacter $\phi$ of $(G, G)$ associated to $X$.

(5.2.5). The image $\phi$ normalizes $C$. Suppose that the derived group of $G$ is quasisimple. We have
(a) $\text{Lie}(R_0C) = \bigoplus_{i=1} \text{Lie}(C)(\phi; 2k_i)$

where $1 = k_1 \leq \cdots \leq k_r$ are the exponents of the Weyl group of $G$.
(b) $\dim \text{Lie}(R_0C)(\phi; 2) = 1$. 

Suppose that the elements $X_i \in \text{Lie}(G_i)(K)$ are nilpotent for $i = 1, 2$, that $\text{df}(X_1) = X_2$, and that $X_1$ is regular, equivalently that $X_2$ is regular. If $C_1 = C_{G_1}(X_1)$ and $C = C_{G_2}(X_2)$, then $C_1 = f^{-1}C$. In particular, $f$ restricts to a surjective separable mapping $f_{|C_1}: C_1 \to C$. 

**Proof.** As before, the assertion is geometric; thus we may and will suppose that $K$ is algebraically closed for the proof. We only must argue that $(\ast) \quad C_1 = f^{-1}C$. Indeed, the remaining assertions follow from $(\ast)$ by using (2.11.1)(d) and the smoothness of $C_1$ (3.4.1).
We will argue that $f|_{C_1} : C_1 \to C$ is surjective; assertion (**) will then follow since $\ker f$ is central in $G_1$. Recall that $C_1 = \zeta_{G_1} \cdot R_u C_1$ and $C = \zeta_{G_2} \cdot R_u C$. The restriction $f|_{C_2} : \zeta_{G_1} \to \zeta_{G_2}$ is surjective (5.3.1).

It remains to argue that $f|_{R_u C_1}$ yields a surjective mapping $R_u C_1 \to R_u C$. Since $G_1$ and $G_2$ are $D$-standard, the centralizers $C_1$ and $C$ are smooth by (3.4.1). Thus the unipotent radicals of $C_1$ and of $C$ are smooth group schemes over $K$. So the surjectivity of $f|_{R_u C_1}$ holds if we only prove that $df : \text{Lie}(R_u C_1) \to \text{Lie}(R_u C)$ is surjective.

But $df|_{\text{Lie}(R_u C_1)}$ is injective since $\ker df$ is central. Moreover, $\dim R_u C_1$ is the semisimple rank of $G_1$, and $\dim R_u C$ is the semisimple rank of $G_2$. Since $f$ is surjective with central kernel, the semisimple ranks of $G_1$ and $G_2$ coincide. Thus $df|_{\text{Lie}(R_u C_1)}$ is bijective and the assertion follows. □

5.4. The normalizer of $C$

Let us again fix a regular nilpotent element $X$ together with a cocharacter $\phi$ associated to $X$. Let $N = N_G(C)$ be the normalizer of $C$.

We will argue in (5.4.2) below that $N$ is a smooth group scheme over $K$. Meanwhile, we consider in the next assertion the $N$-orbit of $X$. Viewing this orbit as a subspace of $\text{Lie}(R_u C)$, we may consider its closure; that closure has a unique structure of reduced subscheme [10, Prop. 2.4.2]. Since the orbit of $X$ is open in its closure, that orbit inherits a structure as a reduced subscheme.

The following argument essentially just records observations made by Serre in his note found in [2, Appendix].

(5.4.1). (a) The $N$-orbit of $X$ is the open subset of $\text{Lie}(R_u C)$ consisting of the regular elements; i.e.

$$\text{Ad}(N)X = \text{Lie}(R_u C)_{\text{reg}}.$$  

(b) The group $N/C$ is connected and has dimension equal to the semisimple rank $r$ of $G$.

(c) In particular, $\dim N = 2r + \dim \zeta_G$.

**Proof.** Before giving the proof, we recall that (**) $C = C^o \cdot \zeta_G$ where $\zeta_G$ is the center of $G$; see (5.2.4).

For the proof of (a), we have evidently $\text{Ad}(N)X \subset \text{Lie}(R_u C)_{\text{reg}}$. Since $\text{Ad}(N)X$ is a reduced scheme, to prove equality it suffices to show that any closed point of $\text{Lie}(R_u C)_{\text{reg}}$ is contained in this orbit. If $K_{\text{alg}}$ is an algebraic closure of $K$ and $Y \in \text{Lie}(R_u C)_{\text{reg}}(K_{\text{alg}})$, then $Y$ is a Richardson element for $B$, where $B$ is the Borel subgroup as in (5.2.2). Since the Richardson elements form a single orbit under $B$, there is $x \in B(K_{\text{alg}})$ for which $\text{Ad}(x)Y = X$. Since $C$ is commutative, a dimension argument shows that $C^o(Y) = C^o$. Since also $C^o = C^o \cdot \zeta_G$; it follows from (**) that $C = C^o \cdot \zeta_G$. Since $xC^o\zeta_G^{-1} = xC^o \cdot (\text{Ad}(x)Y) = C^o \cdot \zeta_G \cdot (X)$, one sees that $x \in N(K_{\text{alg}})$. It follows that $\text{Ad}(N)X = \text{Lie}(R_u C)_{\text{reg}}$.

For (b), first suppose that $K = K_{\text{alg}}$ is algebraically closed. By (a), $(N/C)_{\text{red}}$ identifies with $\text{Lie}(R_u C)_{\text{reg}}$, an open subvariety of the affine space $\text{Lie}(R_u C)$. It follows that $(N/C)_{\text{red}}$ is an irreducible variety; thus the variety $N/C$ is connected.

But then relaxing the assumption on $K$, it follows that $N/C$ is connected in general. Since $\text{Lie}(R_u C)$ has dimension equal to $r$, conclude that $\dim N/C = r$.

Finally, (c) follows since $\dim C = r + \dim \zeta_G$. □

We can now prove:

(5.4.2). $N$ is a smooth subgroup scheme of $G$.

**Proof.** The statement is geometric; thus we may and will suppose $K$ to be algebraically closed. Let $f : G_1 \to G_2$ be a surjective separable morphism with central kernel, and suppose that $G$ is one of the groups $G_1$ or $G_2$.

If $G = G_1$, write $X_1$ for $X$ and set $X_2 = df(X_1)$. If $G = G_2$, write $X_2$ for $X$ and use (5.3.2) to find a regular nilpotent $X_1 \in \text{Lie}(G_1)$ for which $df(X_1) = X_2$.

Now write $C_0 = C_0(X_i)$. It follows from (5.3.3) that $C_1 = f^{-1}C_2$, so we may apply (2.11.2) to see that

(*) $N_{C_1}(C_1)$ is smooth over $K$ if and only if $N_{C_2}(C_2)$ is smooth over $K$.

We are now going to argue: it suffices to prove the result when $G$ is quasisimple in very good characteristic.

Well, if the result is known for quasisimple $G$ in very good characteristic, it follows easily for any semisimple, simply connected group in very good characteristic (since any such group is a direct product of simply connected quasisimple groups). But any semisimple group in very good characteristic is separably isogenous to a simply connected one, so (*) then permits us to deduce the result for any semisimple $G$ in very good characteristic.

For a general $D$-standard group $G$, we must consider a reductive group $H$ of the form $H = H_1 \times T$ where $H_1$ is semisimple in very good characteristic, together with a diagonalizable subgroup scheme $D \subset H$. We suppose that $G$ is separable isogenous to $C_H(D)^0$. The above arguments show that the desired result holds for $H$, and we want to deduce the result for $G$. Again using (*), we may suppose that $G = C_H(D)^0$.

But if $N = N_{C_1}(C_1)$, we see that $N = N_{H_1}(C_H(X))$. Our assumption means that $N_{H_1}(C_H(X))$ is smooth. But then [22, Exp. XI, Cor. 5.3] shows that $N = N_{H_1}(C_H(X))^0$ is smooth, as required.

Thus, we now suppose $G$ to be quasisimple in very good characteristic. Now, $\dim N = 2r$ by (5.4.1), where $r$ is the rank of $G$. Thus to show that $N$ is smooth, we must show that $2r = \dim \text{Lie}(N)$. Since one has always $\dim \text{Lie}(N) \geq \dim N$, it is enough to argue that $\dim \text{Lie}(N) \leq 2r$. 


Write \( n = \{ Y \in g \mid [Y, \text{Lie} C] \subset \text{Lie} C \} \) for the normalizer in \( g \) of \( \text{Lie}(C) \). Evidently \( \text{Lie}(N) \subset n \); it therefore suffices to show that \( \dim n \leq 2r \).

Suppose that \( Y \in n \). Since \( C \) is commutative, evidently \([Y, X], X] = 0\), so that \( Y \in \ker(\text{ad}(X)^2) \). Thus, it suffices to show that

\[
(*) \quad \dim \ker(\text{ad}(X)^2) = 2r.
\]

But in view of our assumptions on the characteristic of \( K \), \((*)\) follows from [23, Cor. 2.5 and Theorem 2.6]. \( \square \)

(5.4.3) \( N \) is a solvable group.

**Proof.** Let \( B \) be the unique Borel subgroup of \( G \) with \( X \in \text{Lie}(B) \) as in (5.2.2). Since \( B \) is solvable, the result will follow if we argue that \( N \subset B \).

Since \( N \) is smooth – in particular, reduced – it suffices to argue that \( B \) contains each closed point of \( N \). Thus, it is enough to suppose that \( K \) is algebraically closed and prove that \( N(K) \subset B(K) \).

Recall first that according to (5.2.1), we have \( C \subset B \). If \( y \in N(K) \) it follows that \( \text{Int}(y)B \) contains \( C \), hence \( \text{Lie}(\text{Int}(y)B) \) contains \( X \). This proves that \( \text{Int}(y)B = B \), so \( y \) normalizes \( B \). Since Borel subgroups are self-normalizing, we deduce \( N(K) \subset B(K) \), and the result follows. \( \square \)

(5.4.4) Write \( S \) for the image of \( \phi \) and write \( \zeta_0^G \) for the connected center of \( G \). Then \( S \cdot \zeta_0^G \) is a maximal torus of \( N \).

**Proof.** Let \( T \subset N \) be any maximal torus of \( N \) containing \( S \). Since \( T \) commutes with the image of \( \phi \), it follows that the space \( \text{Lie}(C)(\phi; 2) \) is stable under \( T \). But that space is one dimensional (5.2.5) and has \( X \) as a basis vector; it follows that \( X \) is a weight vector for \( T \) so that \( T \) lies in the stabilizer in \( G \) of the line \([X] \in \text{P}(\text{Lie}(G)) \). We know by (5.2.4) that \( \zeta_0^G \) is a maximal torus of \( C \); applying [12, 2.10 Lemma and Remark], one deduces that \( S \cdot \zeta_0^G \) is a maximal torus of that stabilizer, which completes the proof. \( \square \)

Note that together (5.4.1), (5.4.3) and (5.4.4) yield Theorem B from the introduction.

(5.4.5) Consider the line \([X] \in \text{P}(\text{Lie}(R_0C)) \) and let \( \Theta \) be the \( N \)-orbit of \([X]\).

(a) The orbit mapping \((a \mapsto [\text{Ad}(a)X]) : N \to \Theta \) is smooth.

(b) The stabilizer \( \text{Stab}_N([X]) \) of \([X] \) in \( N \) is smooth and is equal to \( S \cdot C \).

(c) The \( N \)-orbit of \([X] \) is open and dense in \( \text{P}(\text{Lie}(R_0C)) \).

**Proof.** Recall that a mapping \( f : X \to Y \) between smooth varieties over \( K \) is smooth if the tangent map \( df_x \) is surjective for all closed points of \( X \). If \( X \) and \( Y \) are homogeneous spaces for an algebraic group, it suffices to check that \( df_x \) is surjective for one point \( x \) of \( X \).

Moreover, it follows from [6, Prop. 12.1.2] that if an algebraic group \( H \) acts on a variety \( X \), and if \( x \in X \) is a closed point, then the stabilizer \( \text{Stab}_H(x) \) is a smooth subgroup scheme if and only if the orbit mapping \( H \to H.x \) determined by \( x \) is a smooth morphism.

Now, assertion (a) is the content of [18, Lemma 23] As for (b), first note that the fact that the orbit mapping \( N \to \Theta \) is smooth shows that stabilizer \( \text{Stab}_N([X]) \) is smooth over \( K \). Now, according to [12, 2.10] the stabilizer in \( G \) of the line \([X] \) is \( S \cdot C \). Since \( S \cdot C \) is a closed subgroup of \( N \), the remaining assertion of (b) follows.

For (c), notice that \( \dim \text{N}/(S \cdot C) = \dim \text{N}/C - 1 = r - 1 \) by (5.4.1). Since we have also \( \dim \text{P}(\text{Lie}(R_0C)) = r - 1 \), it follows that the \( N \)-orbit of \([X] \) is open and dense in \( \text{P}(\text{Lie}(R_0C)) \), as required. \( \square \)

Let us write \( D = \text{Stab}_N([X]) = S \cdot C \), and let \( \text{L} \) be the closed point of \( \text{N}/\text{D} \) determined by the trivial coset of \( D \) in \( N \). From the adjoint action of the torus \( S \) on \( \text{Lie}(N) \) one deduces an action of \( S \) on the tangent space \( T_1(N/D) \); thus one may speak of the weight spaces \( T_1(N/D)(\phi; j) \) for \( j \in \text{Z} \).

(5.4.6) Assume that the derived group of \( G \) is quasisimple, and let the positive integers \( k_1, k_2, \ldots, k_r \) be as in 5.1. Then we have the following:

\[
T_1(N/D) = \bigoplus_{i=2}^r T_1(N/D)(\phi; 2k_i - 2).
\]

**Proof.** Let \( \Theta \subset \text{P}(\text{Lie}(R_0C)) \) be the \( N \)-orbit of \([X]\). By (5.4.5)(c), one knows that \( \Theta \) is an open subset of \( \text{P}(\text{Lie}(R_0C)) \); in particular, \( T_{[X]}\Theta = T_{[X]}\text{P}(\text{Lie}(R_0C)) \). Also by (5.4.5)(c), one knows that the orbit mapping \( \alpha : N \to \Theta \) given by \( \alpha(y) = [\text{Ad}(y)X] \) induces an \( S \)-equivariant isomorphism \( \bar{\alpha} : N/D \to \Theta \). Since \( \bar{\alpha}(1) = [X] \), the tangent map to \( \bar{\alpha} \) at \( 1 \) yields an \( S \)-isomorphism between \( T_1(N/D) \) and \( T_{[X]}\Theta = T_{[X]}\text{P}(\text{Lie}(R_0C)) \). The assertion now follows from (5.2.5) and the description of the \( S \)-module structure on the tangent space \( T_{[X]}\text{P}(\text{Lie}(R_0C)) \) given in (2.10.1). \( \square \)

We can now complete the proofs of Theorems C and D from the introduction.

7 Alternatively, one can argue as follows. Write \( \mathcal{L} \) for the tautological line bundle – corresponding to the invertible sheaf \( \mathcal{O}_{\text{P}(\text{Lie}(R_0C))}(-1) \) over \( \text{P}(\text{Lie}(R_0C)) \). Then \( \text{Lie}(R_0C) \setminus \{0\} \) identifies with the total space of \( \mathcal{L} \) with the zero-section removed. It follows that the natural mapping \( \text{Lie}(R_0C) \setminus \{0\} \to \text{P}(\text{Lie}(R_0C)) \) is flat and hence open.
Proof of Theorem C. Consider the quotient morphism
\[ \Phi : N/C \to N/(S \cdot C) = N/D \]
and again write 1 for the closed point of \( N/C \) determined by the trivial coset, and 0 for the closed point of \( N/D \) determined by the trivial coset. Then differentiating \( \Phi \) gives an \( S \)-equivariant mapping
\[ d\Phi_1 : T_1(N/C) \to T_1(N/D). \]
Evidently the kernel of \( d\Phi_1 \) is the image of Lie(S) in \( T_1(N/C) \). Regard \( T_1(N/C) \) as an \( S \)-module; by complete reducibility one can find an \( S \)-subrepresentation \( V \subset T_1(N/C) \) which is a complement to ker \( d\Phi_1 \). Then evidently \( d\Phi_1 \) yields an isomorphism between \( V \) and \( T_1(N/D) \), and the assertion of Theorem C follows. 

Proof of Theorem D. We must argue that \( R_0N \) is defined over \( K \) and split. Keep the preceding notations of this section; in particular, \( S \) is the image of the cocharacter \( \phi \) associated to the regular nilpotent element \( X \in \text{Lie}(G) \). According to Theorem 2.9, it will suffice to show that \( \text{Lie}(S) = \text{Lie}(N)^S \) and that each non-zero weight of \( S \) on \( \text{Lie}(N) \) is positive. It suffices to prove these statements after extending scalars; thus we may and will suppose that \( K \) is algebraically closed.

If \( G \) is any \( D \)-standard reductive group, we may find \( D \)-standard groups \( M_1, \ldots, M_d \) together with a homomorphism \( \Phi : M \to G \) where \( M = \prod_{i=1}^d M_i \), satisfying (a)-(d) of (3.2.4).

Using (5.3.3) we may find a regular nilpotent element \( X_1 \in \text{Lie}(M) \) such that – writing \( C_1 = C_M(X_1) \) – the restriction \( \Phi|_{C_1} : C_1 \to C = C(X) \) is surjective (and separable). Moreover, we may choose a cocharacter \( \phi_1 : G_m \to M \) associated with \( X_1 \) such that \( \phi = \phi \circ \phi_1 \) is associated with \( X \). Write \( S_1 \subset M \) for the image of \( \phi_1 \) and \( S \subset G \) for the image of \( \phi \).

Now, by (3.2.4)(a) each \( M_i \) has a quasisimple derived group. In the case where \( M \) itself has a quasisimple derived group – i.e. if \( M = M_1 \) – one uses (5.2.5) and Theorem C to deduce that
\begin{enumerate}
  \item \( \text{Lie}(S_1) = \text{Lie}(N)^{S_1} \), and
  \item \( \text{the non-zero weights of} \ S_1 \text{ on} \ \text{Lie}(N_1) \text{ are positive,} \)
\end{enumerate}
where we have written \( N_1 = N_M(C_1) \). Since in general \( M \) is a direct product of reductive groups each having a quasisimple derived group, one sees readily that (i) and (ii) hold for \( M \).

The normalizer \( N_1 = N_M(C_1) \) is smooth by Theorem B. Since \( \Phi \) is separable, it follows from (2.11.2) that \( \Phi|_{N_1} : N_1 \to N \) is surjective and separable – i.e. \( d\Phi|_{N_1} : \text{Lie}(N_1) \to \text{Lie}(N) \) is surjective. Using the fact that (i) and (ii) hold together with the surjectivity of \( d\Phi|_{N_1} \), one sees that \( \text{Lie}(S) = \text{Lie}(N)^S \) and that the non-zero weights of \( S \) on \( \text{Lie}(N) \) are positive, and the proof is complete. 

5.5. The tangent map to a Springer isomorphism

In this section, we give the proof of Theorem E. Thus we suppose in this section that the derived group of \( G \) is quasisimple.

We fix a Springer isomorphism \( \sigma : \mathcal{X} \to \mathcal{U} \), and we write \( u = \sigma(X) \) where \( u \in G \) is regular unipotent and \( X \in g \) is regular nilpotent.

Since \( \sigma \) is \( G \)-equivariant, one knows that \( C = C_G(X) = C_G(u) \).

(5.5.1) The restriction of \( \sigma \) to \( \text{Lie} R_0C \) determines an isomorphism \( \gamma : \text{Lie} R_0C \to R_0C \). In particular, the tangent mapping \( d\gamma = (d\gamma)_0 \) determines an isomorphism \( d\gamma : \text{Lie} R_0C \to \text{Lie} R_0C \).

Proof. Indeed, recall that \( C \) is a smooth group scheme, and that \( C = C_G \cdot R_0C \) by (5.2.4), so that \( R_0C \) is the space of fixed points of \( \text{Int}(u) \) on \( \mathcal{U} \) and \( \text{Lie} R_0C \) is the space of fixed points of \( \text{Ad}(u) \) on \( \mathcal{X} \); the assertion is now immediate.

Write \( V = \text{Lie} R_0C \). Then \( d\gamma \) is an endomorphism of \( V \) as an \( N \)-module, where \( N \) is the normalizer in \( G \) of \( C \). As in Section 5.4, we fix a cocharacter \( \phi \) associated to \( X \); write \( S \subset N \) for the image of \( \phi \). We now give the following proof.

Proof of Theorem E. For (1), note first that the mapping \( \gamma \) is in particular an \( S \)-module endomorphism of \( V \). Since \( \dim V(\phi; 2) = 1 \) by Theorem (5.2.5), one knows that \( X \) spans \( V(\phi; 2) \). It follows that \( d\gamma(X) = \alpha X \) for some \( \alpha \in K^\times \).

If now \( Y \in V_{\text{reg}} \) (\( \text{Lie} R_0(C) \))_{\text{reg}} \subset \text{Lie} R_0C \) so that indeed \( d\gamma = \alpha \cdot 1_V \).

For (2), recall that \( B \) is a Borel subgroup of \( G \) with unipotent radical \( U \). That \( \sigma|_{\text{Lie} U} \) is an isomorphism onto \( U \) follows from [2, Remark 10].

Now fix a Richardson element \( X \in \text{Lie}(U)(K) \); then \( X \) is a regular nilpotent element of \( g \), and part (1) shows that \( d\sigma|_{\text{Lie} U}(X) = \alpha X \) for some \( \alpha \in K^\times \). If \( Y \in \text{Lie}(U)(K_{\text{alg}}) \) is a second Richardson element, then \( Y = \text{Ad}(g)X \) for \( g \in B(K_{\text{alg}}) \), and it is then clear by the equivariance of \( d(\sigma|_{\text{Lie} U})_0 \) that \( d(\sigma|_{\text{Lie} U})_0(Y) = \alpha Y \). Since the Richardson elements are dense in \( \text{Lie} U \), the result follows. 

Note that Theorem E need not hold when the derived group of \( G \) fails to be quasisimple. Indeed, take for \( G \) the \( D \)-standard group \( G = GL_n \times GL_m \) where \( n, m \geq 2 \). Then \( g = gl_n \oplus gl_m \), and the mapping
\[ (X, Y) \mapsto (1 + \alpha X, 1 + \beta Y) \]
defines a Springer isomorphism \( \sigma \) for any \( \alpha, \beta \in K^\times \). If \( X_0 \in \mathfrak{gl}_n \) and \( Y_0 \in \mathfrak{gl}_m \) are regular nilpotent, then \( X = (X_0, Y_0) \in \mathfrak{g} \) is regular nilpotent; the mapping \( d\sigma \) has eigenvalues \( \alpha \) and \( \beta \) on \( \text{Lie} R_{\alpha, \beta}(X) \) and hence is not a multiple of the identity if \( \alpha \neq \beta \).

We finally conclude with an argument which gives an alternate proof of (b) of Theorem A in case \( G \) has a quasisimple derived group. This argument does not rely on the fact that \( Z(C_1) \) is smooth; on the other hand, in order to make sense of \( Z(C_1)_{\text{red}} \), we are forced to assume \( K \) to be perfect.

(5.5.2). Let \( K \) be perfect, let \( X_1 \in \mathfrak{g}(K) \) be nilpotent, and let \( C_1 = C_G(X_1) \) be its centralizer. Then the rule \( t \mapsto \sigma(tX_1) \) defines a mapping \( \text{Aff}^1 \to Z(C_1)_{\text{red}} \), and \( X_1 = \sigma(tX_1) \) for some \( c \in K^\times \).

Proof. Let \( u = \sigma(X_1) \) and observe that \( C_1 = C_G(u) \) by the \( G \)-equivariance of \( \sigma \), so in particular, \( u \in C_1 \). Then for each \( t \in \text{Aff}^1 \), and for each \( g \in C_1 \), we have

\[
 g \cdot \sigma(tX_1) \cdot g^{-1} = \sigma(t \text{ Ad}(g)X_1) = \sigma(tX_1).
\]

Since \( \text{Aff}^1 \) is reduced, it follows that \( \sigma(tX_1) \) indeed lies in \( Z(C_1)_{\text{red}} \).

The formula for the tangent mapping of \( \Phi \) is now immediate from Theorem E. \( \square \)

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References


